Topological Recursion

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Introduction

Context

- Recursive formula for hyperbolic volumes (Mirzhakani 2004): .
- Recursive formula for random matrices (2006).
- Topological recursion (Eynard, Orantin 2007).
- Topological recursion for Gromov-Witten invariants (Bouchard, Klemm, Marino, Pasquetti 2009).
- Link with Frobenius manifolds and Givental Formalism (Dunin-Barkowski, Orantin, Shadrin, Spitz 2014).
- Topological recursion from Quantum Airy structure (Kontsevich, Soibelman 2017).

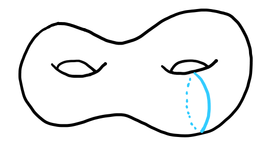
Main Idea

$$\text{Spectral Curve } \Sigma \xrightarrow{\quad\text{Topological Recursion}\quad} \text{Invariants } \omega_{g,n}(\Sigma).$$

Definitions

Riemann Surfaces

A **Riemann Surface** Σ is a connected space of complex dimension one (e.g. $\mathbb{C}, \mathbb{CP}^1, \mathbb{T}, \ldots$).



A map $x\colon \Sigma_1 \to \Sigma_2$ between Riemann Surfaces is **ramified** at z_0 if $dx(z_0) = 0$. Locally x can be written as

$$z \mapsto z^n$$

for some $n > 1$ which we call **ramification index**.

Spectral Curves and Polydifferentials

Definition

A **spectral curve** $S = (\Sigma, x, y, B)$ consists of:

- a Riemann surface Σ
- a covering map $x \colon \Sigma \to \mathbb{C}$ with set of ramification points $\{r_a\}$.
- a function $y \underset{a}{\sim} \sum_k \tilde{t}_{a,k} x^{k/r_a}$ near each branchpoint
- a polydifferential $B \in \Gamma(\Sigma \times \Sigma \setminus \Delta, \pi_1^*(K_\Sigma) \otimes \pi_2^*(K_\Sigma))$ with vanishing residue. That is

$$B(z_1, z_2) \underset{z_1 \to z_2}{\sim} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{holomorphic}.$$

Definition

A **polydifferential** $\omega_{g,n}(z_1, \dots, z_n)$ is an expression of the form $f(z_1, \dots, z_n) dz_1 \dots dz_n$. In other words, a section of

$$\pi_1^*(K_\Sigma) \otimes \dots \otimes \pi_n^*(K_\Sigma).$$

Initial Data

Assume from now on only simple ramification $z \mapsto z^2$. Let $\sigma(z) = -z$ be the local involution that permutes the two sheets near the ramification point.

Given a spectral curve (Σ, x, y, B) define the following:

- A 1-form $\omega_{0,1}(z) = y(z)dx(z)$
- A polydifferential $\omega_{0,2}(z_1, z_2) = B(z_1, z_2)$
- A recursion kernel

$$K_a(z_0, z) = \frac{-1}{2} \frac{\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_0, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))}.$$

Basic Idea

$$\text{Spectral Curve } \Sigma \quad \xrightarrow{?} \quad \text{Invariants } \omega_{g,n}(\Sigma)$$

Recursive Formula

Definition

Define $\omega_{g,n+1}(z_0, z_1, \ldots, z_n)$ via the following formula:

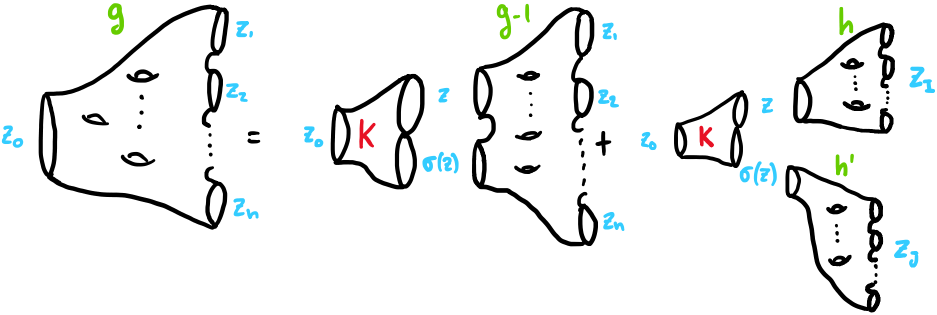
$$\omega_{g,n+1}(z_0, \ldots, z_n) =$$

$$\sum_a \operatorname*{Res}_{z=a} K_a(z_0, z) \left(\omega_{g-1,n+2}(z, \sigma(z), z_1, \ldots, z_n) + \sum_{\substack{h+h'=g \\ I \coprod J = \{z_1, \ldots, z_n\}}} \omega_{h,1+|I|}(z, z_I)\omega_{h',1+|J|}(\sigma(z), z_J) \right)$$

Recursion on the *Euler Characteristic* $\chi(X) = 2 - 2g - n$:

$$\chi(LHS) = 1 - 2g - n < 2 - 2g - n = \chi(RHS).$$

Geometric Interpretation



$$\omega_{g,n+1}(z_0,\dots,z_n) \quad \sim \quad \operatorname*{Res}_{z=a} K_a(z_0,z)\left(\omega_{g-1,n+2} \quad + \quad \omega_{h,1+|I|}(z,z_I)\,\omega_{h',1+|J|}(\sigma(z),z_J))\right).$$

Properties

Properties

The polydifferentials $\omega_{g,n+1}(z_0, z_1, \ldots, z_n)$ satisfy the following properties:

- Symmetric under the action of Σ_n:

$$\omega_{g,n}(z_1, \ldots, z_n) = \omega_{g,n}(z_{\sigma_1}, \ldots, z_{\sigma_n}).$$

- Pole at the ramification points with vanishing residues.
- Symplectic invariance
- Modular invariance

Example

Consider the spectral curve $S = \left(\mathbb{C}, x(z) = z^2/2, y(z) = z, \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right)$.

The brachpoint is at $dx = zdz = 0 \Rightarrow z = 0$ and the local involution is $\sigma(z) = -z$.

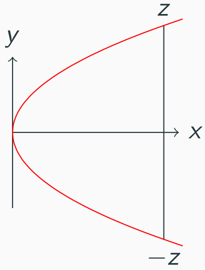


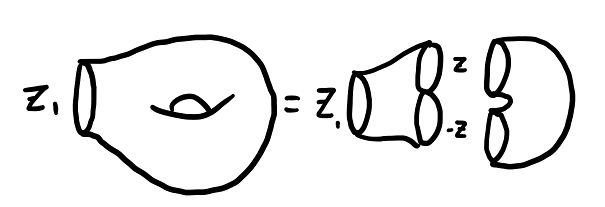
Figure 1: Curve $x = y^2/2$.

In this case we have $\omega_{0,1}(z) = z^2 dz$ and $\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$.

Example

The recursion kernel $K(z_1, z) = \frac{-dz_1}{2z(z_1^2 - z^2)dz}$.

- $\omega_{1,1}(z_1) = \operatorname*{Res}_{z=0} K_0(z_1, z)\omega_{0,2}(z, -z) = \operatorname*{Res}_{z=0} \frac{dz_1 dz}{8z^3(z_1^2 - z^2)} = \frac{dz_1}{8z_1^4}$.



- $\omega_{0,3}(z_1, z_2, z_3) = 2 \operatorname*{Res}_{z=0} K_0(z_1, z)\omega_{0,2}(z, z_2)\omega_{0,2}(-z, z_3) = \frac{dz_1 dz_2 dz_3}{z_1^2 z_2^2 z_3^2}$.

Applications

Kontsevich-Witten Intersection Numbers

The Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of stable curves of genus g and n is constructed by including nodal curves.

Let $\psi_i = c^1(\mathbb{L}_i)$. For $\sum d_i = 3g - 3 + n$ let

$$\langle \tau_{d_1}, \tau_{d_2} \ldots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

Kontsevich-Witten Intersection Numbers

Set

$$W_{g,n}(z_1, \ldots, z_n) = \sum_{d} \langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle_g \prod_{i=1}^{n} \frac{(2d_i - 1)!!}{z_i^{2d_i+2}} dz_i.$$

Theorem

The $W_{g,n}$ are obtained via topological recursion on the spectral curve

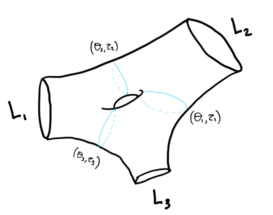
$$\left(\mathbb{C}, x(z) = z^2/2, y(z) = z, B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \right).$$

Hyperbolic Volumes

Fact

Any compact Riemann surface X with $\chi(X) < 0$ admits a (non-unique) hyperbolic metric of constant curvature -1. If we require the boundaries to be geodesics of fixed length then the metric is unique.

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g and n boundary components of lengths $(L_1, \ldots, L_n)$.



It is naturally equipped with Fenchel-Nilsen coordinates (θ_i, τ_i) (lengths and twists).

Hyperbolic Volumes

Although (θ_i, τ_i) are not global coordinates,

$$\omega = \prod d\theta_i \wedge d\tau_i$$

is a well-defined volume form. Consider the volume of the moduli space:

$$\mathcal{V}_{g,n}(L_1, \ldots, L_n) = \int_{\mathcal{M}_{g,n}} \omega.$$

For example we have

$$V_{0,3}(L_1, L_2, L_3) = 1 \qquad V_{1,1}(L) = \frac{1}{48}(L^2 + 4\pi^2).$$

Laplace transform

$$\mathcal{V}_{g,n}(L_1, \ldots, L_n) \longrightarrow W_{g,n}(z_1, \ldots, z_n).$$

For example

$$1 \longrightarrow \frac{1}{z_1^2 z_2^2 z_3^2},$$

$$\frac{1}{48}(L^2 + 4\pi^2) \longrightarrow \frac{1}{8z^4} + \frac{\pi^2}{12z^2}.$$

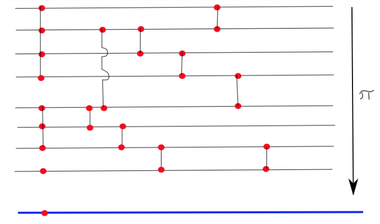
Theorem

The $W_{g,n}$ are obtained via topological recursion on the spectral curve

$$\left(\mathbb{C}, x(z) = z^2, y(z) = \frac{1}{4\pi}\sin(2\pi z), B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right).$$

Hurwitz Numbers

The Hurwitz number $h_{g,n}(k_1, \ldots, k_n)$ is the number of ramified coverings of $\mathbb{CP}^1$ of genus g with $2g - 2 + n$ simple ramification points and one point with ramification profile $\{k_1, \ldots, k_n\}$.



Again we can perform a certain Laplace Transform

$$h_{g,n} \longrightarrow W_{g,n}.$$

Theorem

The $W_{g,n}$ are be obtained via a topological recursion on the spectral curve

$$\left(\mathbb{C}, x(z) = -z + \ln(z), y(z) = z, B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right).$$

Other examples

(From *A short overview of the Topological Recursion*).

Further Questions

What's next?

- Topological recursion for knot invariants (Jones, HOMFLY...).
- Given an enumerative problem, determine if it statisfies the topological recursion.
- Given an enumerative problem, find the spectral curve associated to it.
- Quantum Airy structure analog for higher ramification.

Questions?