Non-Perturbative Topological Recursion and Knot Invariants

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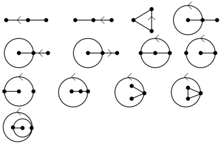
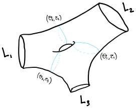
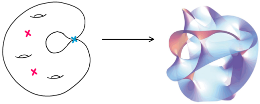
A Big Picture

Topological recursion is an axiomatic construction with links to many interesting geometric problems

- Combinatorics of maps 
- Kontsevich-Witten intersection numbers
- Weil-Peterson volumes 

- Hurwitz Numbers
- Gromov-Witten invariants 

Main Goals

$$\begin{array}{c} \text{Spectral Curve} \\ \text{Complex Curve} \end{array} \xrightarrow{\text{TR}} \begin{array}{c} \text{Correlators } \omega_{g,n} \\ \text{Enumerative Geo.} \end{array} \xrightarrow[]{\substack{\text{Perturbative} \\ \text{Non-Perturbative}}} \begin{array}{c} \text{Wave Function} \\ \text{Quantum Curve} \end{array}$$

- Interpret TR and non-perturbative wave functions as **graph sums**
- Implementation of efficiently algorithm. Compute non-perturbative wave functions and quantum curve in **genus one**
- Application to two conjectures: knot theory and Weierstrass curve

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- Implementation of efficiently algorithm. Compute non-perturbative wave functions and quantum curve in **genus one**
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Warning: if a formula looks too complicated, stare at the picture!

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Topological Recursion and Quantum Curves

Spectral Curves

A **spectral curve** is a tuple $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$, where Σ is a complex curve, x and y are two meromorphic functions on Σ and $\omega_{0,2}$ a fundamental bidifferential.

- **Ramification points** $\mathcal{R} = (p_1, \ldots, p_s)$ of $x\colon \Sigma \to \mathbb{CP}^1$
- Order of ramification r_α at each p_α (simple $r_\alpha = 2$)
- Symmetric **fundamental bidifferential**

$$\omega_{0,2}(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + \text{holo.} \right) dz_1 \, dz_2$$

Topological Recursion

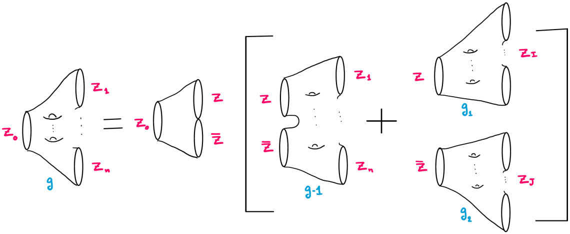
Topological Recursion (Eynard & Orantin 2007)

Symmetric differentials $\omega_{g,n+1}(z_0, z_1, \dots, z_n)$ (**correlators**) defined recursively:

$$\omega_{g,n+1}(z_0, \dots, z_n) :=$$

$$\sum_{p_\alpha \in \mathcal{R}} \operatorname*{Res}_{z=p_\alpha} \frac{\int_o^z \omega_{0,2}(z_1, \cdot)}{(y(z) - y(-z))dx(z)} \left(\omega_{g-1,n+2}(z, \bar{z}, z_1, \dots, z_n) + \sum_{\substack{h+h'=g \\ I \coprod J = \{z_1, \dots, z_n\}}} \omega_{h,1+|I|}(z, z_I)\omega_{h',1+|J|}(\bar{z}, z_J) \right).$$

Recursion on $\chi = 2g - 2 + n$ starting at $\chi = 0$ with $\omega_{0,2}$.



Prototypical Example: Airy Spectral Curve

The **Airy spectral curve** is defined as:

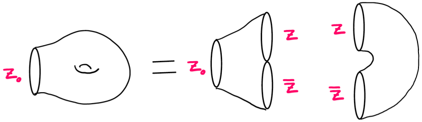
$$\left(\mathbb{CP}^1, x(z) = \frac{z^2}{2}, y(z) = -z, \omega_{0,2} = \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right)$$

- Simple ramification point at $z = 0$
- Recursion kernel

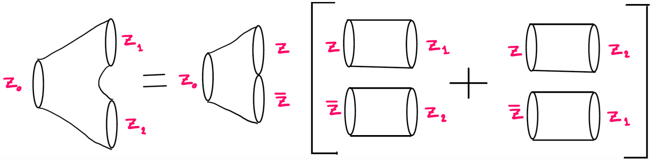
$$K(z_1, z) = \frac{\int_o^z \omega_{0,2}(z_1, \cdot)}{(y(z) - y(-z))dx(z)} = -\frac{dz_1}{2z_1 z(z_1 - z)dz}$$

Prototypical Example: Airy Spectral Curve

Two correlators $\omega_{1,1}$ and $\omega_{0,3}$ with $\chi = 1$:



$$\omega_{1,1}(z_1) = \operatorname*{Res}_{z=0} K(z_1, z)\omega_{0,2}(z, -z) = \operatorname*{Res}_{z=0} \frac{dz_1\, d(-z)}{8z^3 z_1(z_1 - z)} = \frac{dz_1}{8z_1^4}$$



$$\omega_{0,3}(z_1, z_2, z_3) = \operatorname*{Res}_{z=0} K(z_1, z) \left(\omega_{0,2}(z, z_2)\omega_{0,2}(-z, z_3) + \omega_{0,2}(-z, z_2)\omega_{0,2}(z, z_3)\right) = \frac{dz_1\, dz_2\, dz_3}{z_1^2 z_2^2 z_3^2}$$

W-Algebras

Borot, Bouchard, Chidambaram, Creutzig & Noshchenko (2019):

- Reformulate topological recursion in terms of differential operators
- Near a ramification point $p_\alpha \in \mathcal{R}$, coordinate $\frac{\zeta^r}{r} = x - x(p_\alpha)$

$$y = \sum_{\ell \geq r_\alpha} F_{0,1}\left[\begin{smallmatrix}\alpha \\ -\ell\end{smallmatrix}\right]\zeta^{\ell - r_\alpha} \qquad \omega_{0,2} = \left(\frac{\delta_{\alpha_1,\alpha_2}}{(\zeta_1 - \zeta_2)^2} + \sum_{\ell_1,\ell_2 > 0} \phi^{\alpha_1,\alpha_2}_{\ell_1,\ell_2} \zeta_1^{\ell_1 - 1}\zeta_2^{\ell_2 - 1}\right) d\zeta_1 d\zeta_2$$

- Differential operator on functions of variables $\{x^\alpha_\ell\}_{\substack{\alpha \in \mathcal{R} \\ \ell > 0}}$

$$D = \left(\sum_{\alpha,\ell} \frac{F_{0,1}\left[\begin{smallmatrix}\alpha \\ -\ell\end{smallmatrix}\right] + \delta_{\ell,s_\alpha}}{\ell} \partial_{x^\alpha_\ell} + \frac{\hbar}{2} \sum_{\substack{\alpha_1,\alpha_2 \\ \ell_1,\ell_2}} \frac{\phi^{\alpha_1,\alpha_2}_{\ell_1,\ell_2}}{\ell_1 \ell_2} \partial_{x^{\alpha_1}_{\ell_1}} \partial_{x^{\alpha_2}_{\ell_2}}\right)$$

- **Perturbative generating function**

$$Z_{\mathrm{P}}[x^\alpha_\ell, \hbar] := e^D \prod_{\alpha \in \mathcal{R}} Z^{(r_\alpha)}\left[x^\alpha_\ell, \hbar\right]$$

TR from W-Algebras

Building blocks are obtained from W algebra constraints

$$Z_{\mathrm{P}}[x_\ell^\alpha, \hbar] := e^D \prod_{\alpha \in \mathcal{R}} Z^{(r_\alpha)} \left[x_\ell^\alpha, \hbar\right]$$

Simple ramification ($r_\alpha = 2$), then L_i are a rep. of the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n}$:

$$L_{-1} = -\frac{x_1^2}{4\hbar} + \frac{1}{2}\sum_{p_1 \geq 1} (-1)^{p_1} (p_1+2) x_{p_1+2} \partial_{p_1} + \frac{\partial_1}{2}$$

$$L_0 = \frac{1}{2}\sum_{p_1 \geq 1} (-1)^{p_1} p_1 x_{p_1} \partial_{p_1} - \frac{1}{16} + \frac{\partial_3}{2}$$

$$L_{k\geq 1} = \frac{\hbar}{4} \sum_{\substack{p_1+p_2=2(k-1) \\ p_1,p_2\geq 0}} (-1)^{p_1} \partial_{p_1}\partial_{p_2} + \frac{1}{2}\sum_{\substack{p_1-p_2=2(k-1) \\ p_1,p_2\geq 0}} (-1)^{p_1} p_2 x_{p_2} \partial_{p_1} + \frac{\partial_{2k+1}}{2}$$

$$L_i \cdot Z^{(2)}[x_\ell, \hbar] = 0, \forall i \geq -1, \qquad Z^{(2)}[x_\ell, \hbar] = \exp\left(\sum_{n\geq 0, g\geq 0} \frac{\hbar^{g-1}}{n!} F_g^{(r_\alpha)}[x_\ell] \right)$$

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$Z^{(2)}$ generating function of **Kontsevich-Witten** intersection numbers

TR from W-Algebras

- $Z_{\mathsf{P}}[x^\alpha_\ell, \hbar] = \exp\Big(F_{\mathsf{P}}[x^\alpha_\ell, \hbar]\Big)$
- **Free energy** functions $F_{g,n}[x^\alpha_\ell]$ and coefficients $F_{g,n}\Big[\begin{smallmatrix} \alpha_1 & \dots & \alpha_n \\ \ell_1 & \dots & \ell_n \end{smallmatrix}\Big]$:

$$F_{\mathsf{P}}[x^\alpha_\ell, \hbar] =: \sum_{\substack{g \geq 0,\, n \geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\substack{\alpha_1,\dots,\alpha_n \\ \ell_1,\dots,\ell_n}} F_{g,n}\Big[\begin{smallmatrix} \alpha_1 & \dots & \alpha_n \\ \ell_1 & \dots & \ell_n \end{smallmatrix}\Big] \prod_{j=1}^{n} x^{\alpha_j}_{\ell_j} =: \sum_{\substack{g \geq 0,\, n \geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} F_{g,n}[x^\alpha_\ell]$$

- Meromorphic one-forms:

$$d\xi^\alpha_\ell(z) \coloneqq \operatorname*{Res}_{z'=p_\alpha} \left(\int_{p_\alpha}^{z'} \omega_{0,2}(\cdot, z) \right) \frac{d\zeta(z')}{\zeta(z')^{\ell+1}}$$

Theorem (BBCCN)

The correlators $\{\omega_{g,n}\}_{2g-2+n>0}$ can be decomposed as finite sums

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\substack{\alpha_1,\dots,\alpha_n \\ \ell_1,\dots,\ell_n>0}} F_{g,n}\Big[\begin{smallmatrix} \alpha_1 & \dots & \alpha_n \\ \ell_1 & \dots & \ell_n \end{smallmatrix}\Big] \bigotimes_{j=1}^{n} \mathrm{d}\xi^{\alpha_j}_{\ell_j}\big(z_j\big)$$

W-Algebras: Upshot

(i) The TR correlators $\omega_{g,n}$ can be computed from the perturbative generating function $Z_{\mathsf{P}}[x^{\alpha}_{\ell},\hbar]$ via a differential operator:

$$Z_{\mathsf{P}}[x^{\alpha}_{\ell},\hbar] = e^{D} \prod_{\alpha\in\mathcal{R}} Z^{(r_\alpha)}\left[x^{\alpha}_{\ell},\hbar\right]$$

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$Z^{(r_\alpha)}$: intersection numbers in moduli of **higher spin curves** $\overline{M}_{g,n}^{1/r_\alpha}$

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Goal: interpret $Z_{\mathrm{P}} = e^{F_{\mathrm{P}}}$ as a graph sum

Perturbative Graphs

Set of **perturbative graphs** $\mathcal{G}$: $\Gamma = (V, E, L \amalg \tilde{L})$ connected with

- Ordinary L and dilaton $\tilde{L}$ leaves
- Half edges H.
- Vertex labels: genus $g\colon V \to \mathbb{Z}_{\geq 0}$ and ramification points $\alpha\colon V \to \mathcal{R}$
- Half-edge labels: $\ell\colon H \to \mathbb{Z}_{\geq 0}$, such that $\ell(l) > r_\alpha + 1, \forall l \in \tilde{L}$
- Stability

$$\chi_v = 2g(v) - 2 + \hat{\delta}(v) > 0$$

$\hat{\delta}(v)$ valence not counting dilaton leaves

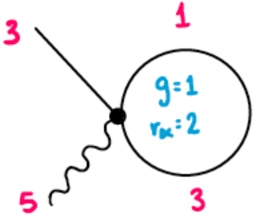
- Euler characteristic

$$\chi(\Gamma) = 2\left(\sum_{v \in V} g(v) + h_1(\Gamma)\right) - 2 + |L|$$

Perturbative Graphs: Weights

- Spectral curve $\mathcal{S}$ determines: $F_{0,1}$, ϕ and $\mathcal{R}$
- Weight $w\colon \mathcal{G} \to \mathbb{C}[\{x_\ell^\alpha\}]$

$$\boxed{w(\Gamma) = \prod_{v\in V}\left(\frac{-1}{F_{0,1}\left[\begin{smallmatrix}\alpha(v)\\ -r_{\alpha(v)}-1\end{smallmatrix}\right]}\right)^{\chi_v} \prod_{i\in L} x_{\ell_i}^{\alpha(\mathfrak{a}(i))} \prod_{j\in\tilde{L}} \frac{-F_{0,1}\left[\begin{smallmatrix}\alpha(\mathfrak{a}(j))\\ -\ell_j\end{smallmatrix}\right]}{F_{0,1}\left[\begin{smallmatrix}\alpha(\mathfrak{a}(j))\\ -r_{\alpha(\mathfrak{a}(j))}-1\end{smallmatrix}\right]} \prod_{e\in E}\phi^{\alpha(\mathfrak{a}(e_1)),\alpha(\mathfrak{a}(e_2))}_{\ell(e_1),\ell(e_2)} \prod_{v\in V} I(v)}$$



$$\left(\left(\frac{-1}{F_{0,1}\left[\begin{smallmatrix}\alpha\\ -3\end{smallmatrix}\right]}\right)^4 \phi^{\alpha,\alpha}_{1,3} F_{0,1}\left[\begin{smallmatrix}\alpha\\ -5\end{smallmatrix}\right] F^{(2)}_{0,4}[1,3,3,5]\right) x_3^\alpha$$

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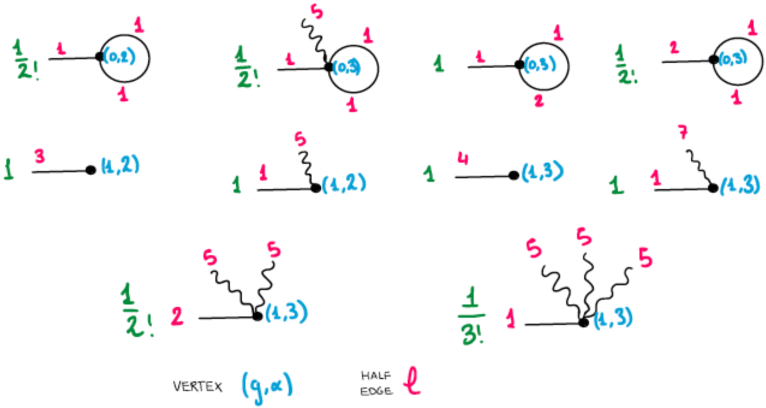
Proposition (Graph Properties)

- Dimensional condition at each vertex $v \in V$

$$\sum \ell(l) = (r_{\alpha(v)} + 1)(2g(v) - 2 + \delta(v))$$

- Weight function $w(\Gamma)$ does not depend on local coordinate ζ_α
- Finitely many graphs of genus g and n ordinary leaves
- $$F_{\mathrm{P}}[x_\ell^\alpha, \hbar] = \sum_{\Gamma\in\mathcal{G}} \hbar^{g(\Gamma)-1}\frac{w(\Gamma)}{|\mathrm{Aut}(\Gamma)|}$$

Perturbative graphs (times automorphism factor) contributing to $F_{1,1}$ for a spectral curve $\mathcal{S}$ with one ramification point of order 2 and one of order 3.



Non-Perturbative TR

From $Z[x_\ell^\alpha, \hbar]$ one can define

- Perturbative wave function ψ_{P}:
    - Quantum Curve Conjecture in $g(\Sigma) = 0$ (Eynard & Bouchard 2015)
    - For $g(\Sigma) > 0$, the wave function ψ_{P}
        - does **not** have modular properties
        - is **not** background independent
        - does not satisfy the quantum curve conjecture (experimental)
- Non-perturbative wave function ψ_{NP} (Eynard & Mariño 2009):
    - Global structure of Σ (compact)
    - Symplectic basis $\{\mathcal{A}_i, \mathcal{B}_i\}_{i \leq g}$ of $H_1(\Sigma, \mathbb{Z})$
    - Parameter ζ_h (=0 in our examples)

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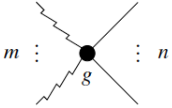
Goal: describe ψ_{NP} as **another** graph sum for genus one curves

Non-Perturbative Graphs

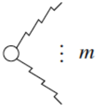
Set of **non-perturbative graphs** $\mathcal{F}$: $\Gamma = (V_{\mathrm{B}} \amalg V_{\mathrm{W}}, E, L)$ connected bipartite with

- Black vertex labels $g \colon V \to \mathbb{Z}_{\geq 0}$
- Black vertices:



$$G_g^{n,(m)}(z) \coloneqq \frac{1}{(2\pi i)^m} \underbrace{\int_o^z \cdots \int_o^z}_{n} \underbrace{\oint_{\mathcal{B}} \cdots \oint_{\mathcal{B}}}_{m} \omega_{g,n+m}$$

- White vertices: choice of theta function $\Theta(\eta) \coloneqq \theta\left[\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}\right](\eta;\tau)$



$$\frac{d^m}{d\eta^m} \ln \Theta(\eta)\bigg|_{\eta = z-o}$$

- Stability of black vertices:

$$2g(v) - 2 + \delta(v) > 0$$

Non-Perturbative Wave Function

Definition

$S(z)$ is a sum over non-perturbative graphs $\mathcal{F}$

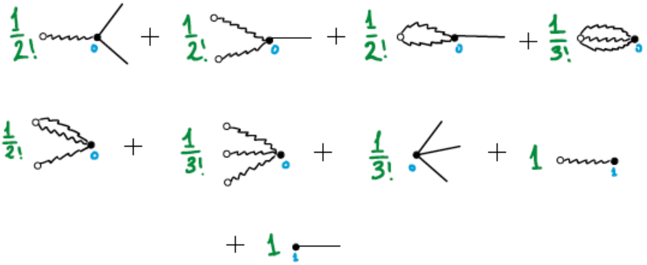
$$S(z) := \frac{1}{h}\sum_{n=2}^{\infty} h^n S_n(z) = \sum_{\Gamma\in\mathcal{F}} h^{\chi(\Gamma)} \frac{w(\Gamma)}{|\mathrm{Aut}(\Gamma)|}$$

Non-perturbative wave function is defined as

$$\psi_{\mathrm{NP}}(z) := \exp\left(\frac{1}{h}\sum_{n=2}^{\infty} h^n S_n(z)\right)$$

- $[k|k]$-kernels defined in a similar way

Non-perturbative graphs contributing to S_2, and the corresponding automorphism factors



Quantum Curves

A **Quantum Curve** $\hat{P}$ of a spectral curve $\mathcal{S}$ with defining polynomial P is a differential operator of the form

$$\hat{P}(\hat{x},\hat{y};h) = P(\hat{x},\hat{y}) + \sum_{n\geq 1} h^n P_n(\hat{x},\hat{y}), \qquad \hat{x} = x, \qquad \hat{y} = h\frac{d}{dx}$$

where P_n are polynomials in $\hat{x}$ and $\hat{y}$ of degree in $\hat{y}$ smaller than P.

Quantum Curve Conjecture

Let $\mathcal{S}$ be a global spectral curve of genus $g > 0$ defined as the zero locus in $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$ of a polynomial $P(x, y)$. Then there exists a quantum curve $\hat{P}(\hat{x},\hat{y};h)$ that annihilates the non-perturbative wave function ψ_{NP} of $\mathcal{S}$, for a suitable choice of half characteristic (ν,μ) and a base point $o \in \Sigma$.

$$P(\hat{x},\hat{y};h)\cdot\psi_{\mathrm{NP}} = 0$$

TR & Graph Sums: Upshot

Given a global spectral curve $\mathcal{S}$:

(i) Perturbative graph sum to obtain correlators $\omega_{g,n}$

(ii) Non-perturbative graph sum to obtain wave function ψ_{NP}

(iii) Can evaluate corresponding quantum curve

Next goal: implement algorithm and find applications

Knot Invariants

Physics Perspective

Chern-Simons (CS) on a compact oriented 3-fold M with gauge group G.

- $G = \mathrm{SU}(2)$: N-colored Jones polynomial $J_K(N; q)$ (Witten 1993)
- $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$: Hyperbolic geometry, A-polynomial

Volume Conjecture (Kashaev 1998, Murakami & Murakami 1999)

Let K by any knot. The N-colored Jones Polynomials $J_K(N, q)$ satisfy

$$\lim_{N\to\infty} \frac{2\pi}{N} \log \left| J_K\left(N; q = e^{\frac{2\pi i}{N}}\right) \right| = \mathrm{Vol}(S^3 \setminus K)$$

Colored Jones vs A-polynomial

- Generalized Volume Conjecture (Gukov 2005): asymptotic expansion of the colored Jones polynomial depending on a parameter $u = N \cdot h$ as $N \to \infty$ and $h \to 0$

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- AJ Conjecture (Garoufalidis 2004): difference equation for the colored Jones polynomial

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- State Integral Model (Hikami 2006): another graph sum from a state model for hyperbolic 3-manifolds

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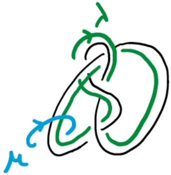
- State Integral Model (Hikami 2006): another graph sum from a state model for hyperbolic 3-manifolds
- (Dimofte, Gukov, Lenells & Zagier 2009): equivalent methods to compute the partition function $G_{\mathbb{C}}$ Chern-Simons

Colored Jones vs A-polynomial

- Generalized Volume Conjecture (Gukov 2005): asymptotic expansion of the colored Jones polynomial depending on a parameter $u = N \cdot h$ as $N \to \infty$ and $h \to 0$
- AJ Conjecture (Garoufalidis 2004): difference equation for the colored Jones polynomial
- State Integral Model (Hikami 2006): another graph sum from a state model for hyperbolic 3-manifolds
- (Dimofte, Gukov, Lenells & Zagier 2009): equivalent methods to compute the partition function $G_{\mathbb{C}}$ Chern-Simons
- $\hat{A}$ Topological Recursion Conjecture
(Dijkgraaf, Fuji & Manabe 2011 and Borot & Eynard 2015)

A-polynomial



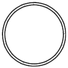
- Hyperbolic 3-manifold M with boundary $\partial M \cong \mathbb{T}^2$ (maybe $S^3 \setminus K$)
- Representation variety $R(\pi_1(M)) = \mathrm{Hom}(\pi_1(K), \mathrm{SL}(2,\mathbb{C}))$
- Generators μ, λ of $\pi_1(\partial M) \cong \mathbb{Z} \times \mathbb{Z}$.
- $R_U \subset R(\pi_1(M))$ upper triangular

$$\rho(\mu) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}, \rho(\lambda) = \begin{pmatrix} \ell & * \\ 0 & \ell^{-1} \end{pmatrix}$$

- **A-polynomial**: defining polynomial of the image $\rho \mapsto (m, \ell)$ in $\mathbb{C}^2$

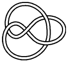
A-polynomial

- Polynomial in two variables $A(x,\ell)$
- Involution $x^r \ell^s A(1/x, 1/\ell) = A(x,\ell)$
  - Unknot $\mathbf{0}_1$:



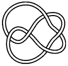
$$\ell - 1$$

  - Figure eight knot $\mathbf{4}_1$:



$$(\ell-1)(\ell^2x^2 - \ell(x^4 - x^3 - 2x^2 - x + 1) + x^2)$$

  - Three twist knot $\mathbf{5}_2$:



$$(\ell-1)\Big(1 + \ell^3x^7 + \ell^2x^2(1 - x + 2x^3 + 2x^4 - x^5) - \ell(1 - 2x - 2x^2 + x^4 - x^5)\Big)$$

A-polynomial Spectral Curve

Spectral curve from A-polynomial

$$\left(\Sigma, \frac{1}{2}\ln(x), \ln(\ell), \omega_{0,2}\right)$$

- Complex curve $\Sigma = \{(x, \ell) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid A(x, \ell) = 0\}$ may be singular
- Normalized bidifferential $\omega_{0,2}$ on resolution $\tilde{\Sigma} \to \Sigma$
- Ramification points related by involution $\iota(x, \ell) = (1/x, 1/\ell)$
- $\zeta_h = 0$ if $g(\Sigma) = 1$ and $\iota_* = -\mathrm{id}$
- Genus ≥ 1 (except $\mathbf{10}_{152}^{(1)}$ up to 8 crossings)

$\hat{A}$-TR Conjecture

Conjecture (Borot & Eynard)

- Let M be hyperbolic 3-fold with $\partial M \cong \mathbb{T}^2$ (maybe $S^3 \setminus K$).
- Chern-Simons partition function (Jones or AJ or Hikami)

$$\mathcal{J}_{\mathrm{CS}} = \exp\left(\sum_{n\geq 0} h^{n-1} S_n\right)$$

- Non-perturbative $[2 \mid 2]$-kernel $\psi_{\mathrm{NP}}^{[2|2]}$ of the A-polynomial spectral curve $A(m,\ell)$ of M. Then the following agree:

$$\mathcal{J}_{\mathrm{CS}}\,(u,h) = C_h e^{\frac{1}{h}S_0 + S_1}\left(\psi_{\mathrm{NP}}^{[2|2]}(m, 1/m)\right)^{1/2}$$

Earlier version by Dijkgraaf, Fuji & Manabe 2011 using perturbative part found discrepancies.

Computations

Algorithm

Given a spectral curve $\mathcal{S} = (\Sigma, x, y)$ (• if genus one)

- Find singularities and genus of Σ and ramifications of $x\colon \Sigma \to \mathbb{CP}^1$
- One-forms $\eta_1, \ldots, \eta_g$ and fundamental bidifferential $\omega_{0,2}$
- Compute elliptic invariants (g_2, g_3)
- Solve $W^i_{\alpha,k} \cdot Z^{(r)} = 0$ to evaluate the building blocks $Z^{(r)}[x^\alpha_\ell, \hbar]$
- Generate perturbative graphs and evaluate $d\xi^\alpha_\ell$ to find $F_{g,n}$ and $\omega_{g,n}$
- Evaluate Θ, $\oint_{\mathcal{B}} d\xi^\alpha_\ell$, $\int_o^p d\xi^\alpha_\ell$ to find $G^{g,(d)}_n$'s
- Generate non-perturbative graphs to find S_n's

Fundamental Forms

Compact curve $\Sigma = \{(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid P(x, y) = 0\}$

- Forms on Σ described by combinatorics of Newton polygon
- Blow-up (if Σ singular)
- Example: knot $\mathbf{4}_1$ $A(x, \ell) = (\ell^2 x^2 - \ell(x^4 - x^3 - 2x^2 - x + 1) + x^2)$

$$\eta(x, \ell) = \frac{(1 - x^2)}{2\ell x^2 - (x^4 - x^3 - 2x^2 - x + 1)} dx$$

$$\omega_{0,2}(x_1, \ell_1, x_2, \ell_2) = \frac{dx_1 dx_2}{12(x_1 - x_2)^2} \frac{1}{A_\ell(x_1, \ell_1) A_\ell(x_2, \ell_2)}$$
$$\Big(12(1 - x_1 - x_2 + x_1^2 x_2 + x_1 x_2^2) - 19(x_1^2 + x_2^2) + 7(x_1^4 + x_2^4) + 2x_1 x_2 + 38 x_1^2 x_2^2$$
$$+ 10(x_1^3 x_2 + x_1 x_2^3) + 12(x_1^3 x_2^2 + x_1^2 x_2^3) - 19(x_1^4 x_2^2 + x_1^2 x_2^4) + 2x_1^3 x_2^3 + 12 x_1^4 x_2^4$$
$$+ 2(x_1^2 x_2^3 \ell_1 + x_1^3 x_2^2 \ell_2 - x_1^2 x_2^4 \ell_1 - x_1^4 x_2^2 \ell_2 + x_1 x_2^2 \ell_2 + x_1^2 x_2 \ell_1) - 12(x_1^2 \ell_1 + x_2^2 \ell_2)$$
$$+ 24(x_1^2 x_2^2 \ell_2 + x_1^2 x_2^2 \ell_1 + x_1^2 x_2^2 \ell_1 \ell_2) - 12(x_1^4 x_2^3 + x_1^3 x_2^4)\Big)$$

Curves of Genus One

- Isomorphism via integration of the unique holomorphic one-form

$$a\colon \Sigma \to \mathbb{C}/\Lambda, \qquad a(p) = \frac{1}{\varpi_1}\int_o^p \eta$$

- Fundamental bidifferential

$$\omega_{0,2}(z_1, z_2) = (\wp(z_1 - z_2) + \kappa) dz_1 dz_2$$

- Evaluation of one-forms $d\xi_\ell^\alpha$

$$d\xi_\ell^\alpha(z) = \frac{1}{\ell!}\frac{d^{\ell-1}}{d\zeta^{\ell-1}}\left(\wp(z - z(\zeta))\frac{dz}{d\zeta}\right)\bigg|_{\zeta=0} dz$$

$$\frac{dz}{d\zeta} = \frac{dz}{dx}\frac{dx}{d\zeta} = \eta(x(\zeta), y(\zeta))\zeta^{r-1}$$

- Relations to theta functions

$$\frac{d}{dz}\ln\left(\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z;\tau)\right) = \zeta(z;\tau) - G_2 z$$

Graph Generation

Graph generation:

- Building blocks can be pre-computed
- Avoid redundancies of differential operator
- Maggiolo-Pagani: efficient generating algorithm (modified and implemented by Greyson Potter)

Table 1: Generation of Perturbative Graphs $|\mathcal{R}| = 4$

(g, n)	Run time (s)	(g, n)	Run time (s)
$(0, 3)$	0.003	$(0, 6)$	2.088
$(1, 1)$	0.003	$(1, 4)$	17.84
$(0, 4)$	0.011	$(2, 2)$	52.57
$(1, 2)$	0.078	$(0, 7)$	48.47
$(0, 5)$	0.103	$(1, 5)$	554.8
$(1, 3)$	0.593	$(2, 3)$	2121
$(2, 1)$	1.249	$(3, 1)$	2404

Graph Generation

Graph generation:

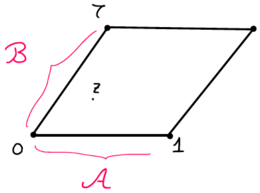
- Building blocks can be pre-computed
- Avoid redundancies of differential operator
- Maggiolo-Pagani: efficient generating algorithm (modified and implemented by Greyson Potter)

Table 1: Number of terms in $F_{g,n}[x_\ell^\alpha]$ for a 1 and 4 simple ramifications

(g, n)	$\lvert\mathcal{R}\rvert = 1$	$\lvert\mathcal{R}\rvert = 4$	(g, n)	$\lvert\mathcal{R}\rvert = 1$	$\lvert\mathcal{R}\rvert = 4$
(0, 3)	1	4	(2, 2)	109	8698
(1, 1)	3	12	(0, 7)	55	4540
(0, 4)	3	18	(1, 5)	122	28796
(1, 2)	9	78	(2, 3)	248	73840
(0, 5)	9	108	(3, 1)	334	76228
(1, 3)	23	540	(0, 8)	122	29606
(2, 1)	42	948	(1, 6)	261	199598
(0, 6)	23	690	(2, 4)	520	575818
(1, 4)	55	3991	(3, 2)	849	793602

Applications

Weierstrass Spectral Curve



The **Weierstrass spectral curve**:

$$\left(\mathbb{C}/\Lambda, x(z) = \wp(z), y(z) = \wp'(z), \left(\wp(z1 - z2) + G_2\right) dz_1 dz_2\right)$$

- With $(g_2, g_3) = (0, 4)$
- If $g_2 = 0$ then $\zeta_h = 0$
- Weierstrass equation $y^2 = 4x^3 - 4$
- Symplectic basis $\mathcal{A} = [0, 1]$ and $\mathcal{B} = [0, \tau]$

Weierstrass Spectral Curve

- $S_0(z) = \int_0^z \wp'(z)^2$
- $S_1(z) = -\frac{\log \wp'(z)}{2}$
- $S_2(z) = \frac{1}{12\wp'(z)^3}(19\wp(z)^2 - 4\wp(z)^5)$
- $S_3(z) = \frac{1}{36\wp'(z)^6}(-2\wp(z)^{10} + 18\wp(z)^7 + 159\wp(z)^4 + 230\wp(z))$
- $S_4(z) = \frac{1}{6480\wp'(z)^9}(-80\wp(z)^{15} + 640\wp(z)^{12} + 9400\wp(z)^9 + 337067\wp(z)^6 + 906596\wp(z)^3 + 88952)$
- $S_5(z) = \frac{1}{648\wp'(z)^{12}}(-2\wp(z)^{20} + 36\wp(z)^{17} - 6\wp(z)^{14} + 12016\wp(z)^{11} + 538839\wp(z)^8 + 2345964\wp(z)^5 + 810118\wp(z)^2)$
- $S_6(z) = \frac{1}{816480\wp'(z)^{15}}(-672\wp(z)^{25} + 21519680\wp(z)^{22} - 159722000\wp(z)^{19} + 515348640\wp(z)^{16} - 682123900\wp(z)^{13} + 15322295767\wp(z)^{10} + 88286246520\wp(z)^7 + 63712854800\wp(z)^4 + 4159251040\wp(z))$
- $S_7(z) = \frac{1}{21870\wp'(z)^{18}}(-5\wp(z)^{30} - 22850913\wp(z)^{27} + 206235312\wp(z)^{24} - 827215278\wp(z)^{21} + 1936629852\wp(z)^{18} - 2769153528\wp(z)^{15} + 12887012988\wp(z)^{12} + 80565969228\wp(z)^9 + 98319454233\wp(z)^6 + 18580283543\wp(z)^3 + 245668068)$

Weierstrass Spectral Curve

- Set $x = \wp(z)$. Solve
$$\hat{P}(\hat{x}, \hat{y}; h) \cdot \psi_{\mathrm{NP}} = 0$$
- Quantum Curve Conjecture up to $O(h^8)$:

Theorem (G.)

$$\begin{aligned}\hat{P}(\hat{x}, \hat{y}; h) =& h^2 \frac{d^2}{dx^2} - 4(x^3 - 1) + h^2 \frac{x}{2^2 3} + h^4 \frac{1}{2^6 3^2} \frac{d}{dx} + h^4 \frac{x^2}{2^8 3^3} \\ &+ h^6 \frac{x}{2^{12} 3^4} \frac{d}{dx} + h^6 \frac{x^3}{2^{14} 3^5} + h^8 \frac{x^2}{2^{18} 3^6} \frac{d}{dx} + O(h^8).\end{aligned}$$

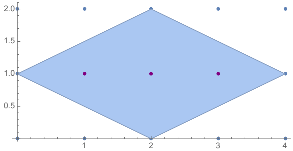
- Closed form:

Conjecture (G.)

$$\hat{P}(\hat{x}, \hat{y}; h) = h^2 \frac{d^2}{dx^2} - 4(x^3 - 1) + \frac{h^2}{576 - h^2 x} \left(48x + h^2 \frac{d}{dx} \right)$$

Figure Eight 4_1

- A-polynomial $A_{4_1}(x,\ell) = \ell^2 x^2 - \ell(x^4 - x^3 - 2x^2 - x + 1) + x^2$
- Two singular points $(1,-1)$ and $(-1,1)$
- Newton Polygon



- Four simple ramification points $(x,\ell) = \left(\frac{3\pm\sqrt{5}}{2}, 1\right), \left(\frac{-1\pm i\sqrt{3}}{2}, -1\right)$

Figure Eight 4_1

Theorem (G., Potter)

Let $\psi_{\text{NP}}^{[2|2]}$ be the non-perturbative [2|2]-kernel obtained from topological recursion on the A-polynomial of the figure right knot $\mathbf{4}_1$, for the choice of half-characteristic $(\nu, \mu) = \left(0, \frac{1}{2}\right)$ and basepoint $o = \left(\frac{3+\sqrt{5}}{2}, 1\right)$. There is an agreement

$$\left(\psi_{\text{NP}}^{[2|2]}\right)^{\frac{1}{2}} = \mathcal{J}_{\text{CS}}$$

where $\mathcal{J}_{\text{CS}}$ denotes the perturbative function of $\mathbf{4}_1$ up to order $O(h^7)$.

- AJ Conjecture (Dimofte,Gukov,Lenells & Zagier 2009)
- Hikami state integral model (Dijkgraaf, Fuji & Manabe 2011)

Figure Eight 4_1

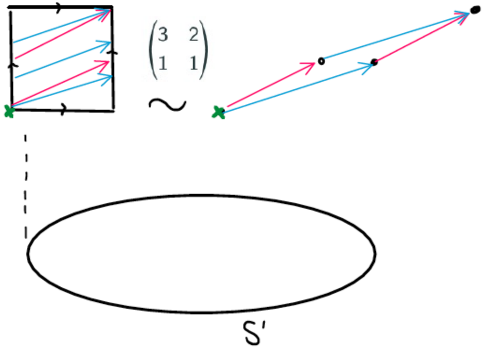
$$G_g^{n,(m)}(x) := \frac{1}{(2\pi i)^m} \underbrace{\int_o^z \cdots \int_o^z}_{n} \underbrace{\oint_{\mathcal{B}} \cdots \oint_{\mathcal{B}}}_{m} \omega_{g,n+m}$$

- $G_0^{0,(6)} = \frac{1122304}{307546875}$
- $G_2^{0,(4)} = \frac{2}{20503125\Delta(x)^8}(-234752x^{16} + 2564703x^{15} - 8808220x^{14} + 6090610x^{13} + 5770000x^{12} + 72992081x^{11} - 163015744x^{10} - 24209375x^9 + 281797720x^8 - 24209375x^7 - 163015744x^6 + 72992081x^5 + 5770000x^4 + 6090610x^3 - 8808220x^2 + 2564703x - 234752)$
- $G_4^{0,(2)} = \frac{1}{1366875\Delta(x)^{10}}(12736x^{20} - 263667x^{19} + 1499890x^{18} - 1315655x^{17} - 5030352x^{16} - 1733042x^{15} + 166645716x^{14} - 374143470x^{13} + 99288000x^{12} + 1029625271x^{11} - 1679582106x^{10} + 1029625271x^9 + 99288000x^8 - 374143470x^7 + 166645716x^6 - 1733042x^5 - 5030352x^4 - 1315655x^3 + 1499890x^2 - 263667x + 12736)$
- $G_6^{0,(0)} = \frac{1}{360\Delta(x)^{12}}(3x^{23} - 12x^{22} + 28x^{21} + 1320x^{20} - 5106x^{19} + 6924x^{18} + 38643x^{17} - 152016x^{16} + 222629x^{15} + 33156x^{14} - 560514x^{13} + 862392x^{12} - 560514x^{11} + 33156x^{10} + 222629x^9 - 152016x^8 + 38643x^7 + 6924x^6 - 5106x^5 + 1320x^4 + 28x^3 - 12x^2 + 3x)$
- $G_0^{1,(4)} = \frac{6079232}{307546875}$
- $G_2^{1,(2)} = \frac{1}{6834375\Delta(x)^{10}}(-371936x^{20} + 4803226x^{19} - 21523757x^{18} + 38988643x^{17} - 35114069x^{16} + 161440552x^{15} + 107735247x^{14} - 1577876619x^{13} + 2268678786x^{12} + 2942948147x^{11} - 7615329394x^{10} + 2942948147x^9 + 2268678786x^8 - 1577876619x^7 + 107735247x^6 + 161440552x^5 - 35114069x^4 + 38988643x^3 - 21523757x^2 + 4803226x - 371936)$
- $G_4^{1,(0)} = \frac{1}{32805000\Delta(x)^{12}}(85072x^{24} - 2099108x^{23} + 13455756x^{22} - 12921090x^{21} + 147170281x^{20} - 698849930x^{19} + 3524017584x^{18} + 2230020768x^{17} - 34162937438x^{16} + 79999579944x^{15} - 26758597924x^{14} - 155011998246x^{13} + 270985521139x^{12} - 155011998246x^{11} - 26758597924x^{10} + 79999579944x^9 - 34162937438x^8 + 2230020768x^7 + 3524017584x^6 - 698849930x^5 + 147170281x^4 - 12921090x^3 + 13455756x^2 - 2099108x + 85072)$
- $G_0^{2,(2)} = \frac{27617936}{922640625}$

Non-perturbative graph sum:

- $S_2(x) = \frac{x^6-x^5-2x^4+15x^3-2x^2-x+1}{6\Delta(x)^3}$
- $S_3(x) = \frac{4x^3\left(x^6-x^5-2x^4+5x^3-2x^2-x+1\right)}{\Delta(x)^6}$
- $S_4(x) = -\frac{x}{45\Delta(x)^9}(x^{16} - 4x^{15} - 128x^{14} + 36x^{13} + 1074x^{12} - 5630x^{11} + 5782x^{10} + 7484x^9 - 18311x^8 + 7484x^7 + 5782x^6 - 5630x^5 + 1074x^4 + 36x^3 - 128x^2 - 4x + 1)$
- $S_5(x) = \frac{4x^3}{3\Delta(x)^{12}}(x^{18} + 5x^{17} - 35x^{16} + 240x^{15} - 282x^{14} - 978x^{13} + 3914x^{12} - 3496x^{11} - 4205x^{10} + 9819x^9 - 4205x^8 - 3496x^7 + 3914x^6 - 978x^5 - 282x^4 + 240x^3 - 35x^2 + 5x + 1)$
- $S_6(x) = \frac{2x}{945\Delta(x)^{15}}(x^{28} + 2x^{27} + 169x^{26} + 4834x^{25} - 24460x^{24} + 241472x^{23} - 65355x^{22} - 3040056x^{21} + 13729993x^{20} - 15693080x^{19} - 36091774x^{18} + 129092600x^{17} - 103336363x^{16} - 119715716x^{15} + 270785565x^{14} - 119715716x^{13} - 103336363x^{12} + 129092600x^{11} - 36091774x^{10} - 15693080x^9 + 13729993x^8 - 3040056x^7 - 65355x^6 + 241472x^5 - 24460x^4 + 4834x^3 + 169x^2 + 2x + 1)$
- $S_7(x) = \frac{8x^3}{45\Delta(x)^{18}}(x^{30} + 47x^{29} - 176x^{28} + 3373x^{27} + 9683x^{26} - 116636x^{25} + 562249x^{24} - 515145x^{23} - 3761442x^{22} + 14939871x^{21} - 15523117x^{20} - 29061458x^{19} + 96455335x^{18} - 71522261x^{17} - 80929522x^{16} + 179074315x^{15} - 80929522x^{14} - 71522261x^{13} + 96455335x^{12} - 29061458x^{11} - 15523117x^{10} + 1493971x^9 - 3761442x^8 - 515145x^7 + 562249x^6 - 116636x^5 + 9683x^4 + 3373x^3 - 176x^2 + 47x + 1)$

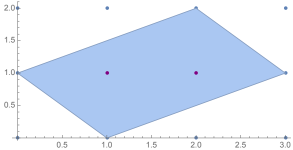
Once Punctured Torus Bundle L^2R



- Torus bundle over S^1: $M = (T^2 \setminus \{0\}) \times [0,1]/(x,0) \sim (\phi(x),1)$
- Monodromy $\phi = L^2R = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$
- Admits complete hyperbolic structure (Thurston)

Once Punctured Torus Bundle L^2R

- A-polynomial $A_{L_2R}(x,\ell) = \ell^2x^2 + \ell(-x^3 + 2x^2 + 2x - 1) + x$.
- One singular point $(1,-1)$
- Newton Polygon



- Four simple ramification points

$$(x,\ell) = \left(\frac{1}{2} + \sqrt{2} \mp \frac{1}{2}\sqrt{5 + 4\sqrt{2}}, \frac{1}{2}\left(1 + \sqrt{2} \pm \sqrt{2\sqrt{2} - 1}\right)\right),$$

$$(x,\ell) = \left(\frac{1}{2} - \sqrt{2} \mp \frac{1}{2}i\sqrt{4\sqrt{2} - 5}, \frac{1}{2}\left(1 - \sqrt{2} \mp i\sqrt{1 + 2\sqrt{2}}\right)\right).$$

Once Punctured Torus Bundle L^2R

Theorem (G., Potter)

Let $\psi_{\mathrm{NP}}^{[2|2]}$ be the non-perturbative [2|2]-kernel obtained from topological recursion on the A-polynomial of L^2R, for the choice of half-characteristic $(\nu, \mu) = \left(0, \frac{1}{2}\right)$ and basepoint $o = \left(\frac{3+\sqrt{5}}{2}, 1\right)$. There is an agreement

$$\left(\psi_{\mathrm{NP}}^{[2|2]}\right)^{\frac{1}{2}} = \mathcal{J}_{\mathrm{CS}}$$

where $\mathcal{J}_{\mathrm{CS}}$ denotes the perturbative function of L^2R up to order $O(h^5)$.

- Hikami state integral model (Dijkgraaf, Fuji & Manabe 2011)

Once Punctured Torus Bundle L^2R

$$G_g^{n,(m)}(x) := \frac{1}{(2\pi i)^m} \underbrace{\int_o^z \cdots \int_o^z}_{n} \underbrace{\oint_{\mathcal{B}} \cdots \oint_{\mathcal{B}}}_{m} \omega_{g,n+m}$$

- $G_1^{0,(4)} = \frac{1}{28812\Delta(x)^5}(-405x^{10}+2509x^9+1829x^8-31810x^7+17655x^6+103891x^5+17655x^4-31810x^3+1829x^2+2509x-405)$
- $G_3^{0,(2)} = \frac{1}{98784\Delta(x)^7}(405x^{14} - 15543x^{13} + 85620x^{12} + 205355x^{11} - 848819x^{10} + 2602758x^9 + 3614417x^8 - 2326433x^7 + 3614417x^6 + 2602758x^5 - 848819x^4 + 205355x^3 + 85620x^2 - 15543x + 405)$
- $G_5^{0,(0)} = \frac{x}{120\Delta(x)^{9/2}}(5x^{16} + 330x^{14} + 345x^{13} + 923x^{12} + 5863x^{11} + 1742x^{10} + 2213x^9 + 12970x^8 + 2213x^7 + 1742x^6 + 5863x^5 + 923x^4 + 345x^3 + 330x^2 + 5)$
- $G_1^{1,(2)} = \frac{1}{76832\Delta(x)^7}(-1629x^{14} + 16263x^{13} - 32316x^{12} - 96499x^{11} + 235411x^{10} + 1561162x^9 - 602369x^8 - 3948791x^7 - 602369x^6 + 1561162x^5 + 235411x^4 - 96499x^3 - 32316x^2 + 16263x - 1629)$
- $G_3^{1,(0)} = \frac{1}{1778112\Delta(x)^{9/2}}(2403x^{18} - 122715x^{17} + 979587x^{16} + 6670552x^{15} + 3536346x^{14} + 85677342x^{13} + 209713378x^{12} - 94163736x^{11} + 43538733x^{10} + 570479611x^9 + 43538733x^8 - 94163736x^7 + 209713378x^6 + 85677342x^5 + 3536346x^4 + 6670552x^3 + 979587x^2 - 122715x + 2403)$
- $G_1^{2,(0)} = \frac{1}{13829760\Delta(x)^{9/2}}(-50175x^{18} + 573019x^{17} - 945107x^{16} - 5002380x^{15} + 43143278x^{14} + 222424934x^{13} + 80209986x^{12} - 626011936x^{11} + 338200043x^{10} + 1765057977x^9 + 338200043x^8 - 626011936x^7 + 80209986x^6 + 222424934x^5 + 43143278x^4 - 5002380x^3 - 945107x^2 + 573019x - 50175)$
- $G_0^{0,(6)} = \frac{9153}{4705960}$
- $G_2^{0,(4)} = \frac{1}{4840416\Delta(x)^4}(-35235x^{16}+468576x^{15}-1040588x^{14}-8404896x^{13}+28367282x^{12}+60969152x^{11}-203134224x^{10}-4676800x^9+482817047x^8-4676800x^7-203134224x^6+60969152x^5+28367282x^4-8404896x^3-1040588x^2+468576x-35235)$

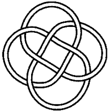
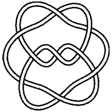
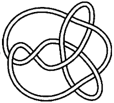
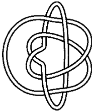
Once Punctured Torus Bundle L^2R

Non-perturbative graph sum:

- $S_2(x) = \frac{5x^6-11x^5+22x^4+105x^3+22x^2-11x+5}{24\Delta(x)^3}$
- $S_3(x) = \frac{-x^{12}+6x^{11}+67x^{10}+466x^9-298x^8-130x^7+1339x^6-130x^5-298x^4+466x^3+67x^2+6x-1}{64\Delta(x)^6}$
- $S_4(x) = \frac{x}{720\Delta(x)^9}(1 - 68x - 3770x^2 + 137x^3 - 30073x^4 - 58605x^5 + 104390x^6 + 20753x^7 - 222062x^8 + 20753x^9 + 104390x^{10} - 58605x^{11} - 30073x^{12} + 137x^{13} - 3770x^{14} - 68x^{15} + x^{16})$
- $S_5(x) = \frac{x^2}{24\Delta^{12}}(1+86x+179x^2+3870x^3+7447x^4-7820x^5+51914x^6+60396x^7-183475x^8-25486x^9+311325x^{10}-25486x^{11}-183475x^{12}+60396x^{13}+51914x^{14} - 7820x^{15} + 7447x^{16} + 3870x^{17} + 179x^{18} + 86x^{19} + x^{20})$

Other Knots

Other knots with A-polynomials of genus one:

- $\mathbf{8}_{18}$ : 6 simple ramifications
- $\mathbf{9}_{35}$ : 2 simple 1 order three
- $\mathbf{9}_{48}$ : 4 simple 1 order three
- $\mathbf{10}_{139}$  : 6 simple

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(Rest of you probably in my thesis acknowledgements)

Gràcies a tothom per ser aquí !!!!

Questions?