Non-Perturbative Topological Recursion with an Application to Knot Invariants

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12-13-2018

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CEO Topological Recursion

Main Idea

$$\text{Spectral Curve } \Sigma \quad \xrightarrow{\text{Topological Recursion}} \quad \text{Invariants } \omega_{g,n}(\Sigma).$$

Spectral Curve

A **spectral curve** is a triple $C = (\Sigma, x, y)$ where Σ is a compact genus g Riemann Surface with a choice of a symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)$, and x and y are two meromorphic functions with distinct critical points.

Assume x and y satisfy a polynomial equation

$$P(x, y) = 0.$$

The **degree** of a spectral curve is the degree of $x\colon \Sigma \to \mathbb{CP}^1$.

The **ramification points** $R = (a_1, \ldots, a_r)$ are points that satisfy $dx(a_i) = 0$. Let $\langle \sigma_a \rangle$ the local deck transformation group.

Fundamental differentials

Definition

The **fundamental bidifferential of second kind** is the unique symmetric bidifferential $B(z_1, z_2)$ with poles of order 2 along the diagonal $z_1 = z_2$ and that it is normalized on $\mathcal{A}_i$-cycles:

$$\oint_{z_1 \in \mathcal{A}_i} B(z_1, z_2) = 0.$$

Definition

Given two points $a, b \in \Sigma$, the **fundamental differential of third kind** is the unique meromorphic one-form $\omega_b^a(z)$ with poles at $z = a$ and $z = b$ and residues 1 and -1 respectively, and that is normalized on the $\mathcal{A}_i$-cycles:

$$\oint_{z \in \mathcal{A}_i} \omega_b^a(z) = 0.$$

Recursive Formula

With this ingredients, set

- Disk amplitude: $\omega_{0,1}(z_1) = y(z_1)dx(z_1)$.
- Cylinder amplitude: $\omega_{0,2}(z_1, z_2) = B(z_1, z_2)$.
- Recursion kernel: $K_a(z_0, z) = \frac{\omega^{z-\alpha}(z_0)}{(y(z)-y(\sigma_a(z)))dx(z)}$.

Definition (simple ramification)

Define the correlators $\boldsymbol{\omega_{g,n+1}(z_0, z_1, \ldots, z_n)}$ via the following formula:

$$\boldsymbol{\omega_{g,n+1}(z_0,\ldots,z_n) =}$$

$$\boldsymbol{\sum_{a\in R} \operatorname*{Res}_{z=a} K_a(z_0, z) \left(\omega_{g-1,n+2}(z, \sigma(z), z_1, \ldots, z_n) + \sum_{\substack{h+h'=g \\ I \coprod J=\{z_1,\ldots,z_n\}}} \omega_{h,1+|I|}(z, z_I)\omega_{h',1+|J|}(\sigma(z), z_J) \right).}$$

Recursion on the *Euler Characteristic* $\chi(X) = 2 - 2g - n$:

$$\chi(LHS) = 1 - 2g - n < 2 - 2g - n = \chi(RHS).$$

Properties

From $\omega_{g,1}(z)$ define the **Free Energies**

$$F_g = \frac{1}{2g-2} \sum_{a \in R} \operatorname*{Res}_{z=a} \Phi(z) \omega_{g,1}(z) \in \mathbb{C}.$$

Properties

The polydifferentials $\omega_{g,n}(z_1, \ldots, z_n)$ satisfy the following properties:

- Symmetry: $\omega_{g,n}(z_1, \ldots, z_n) = \omega_{g,n}(z_{\sigma_1}, \ldots, z_{\sigma_n})$.
- Poles at the ramification points with vanishing residues.
- Symplectic invariance: F_g invariant under Φ preserving $dx \wedge dy$.
- Modular properties: F_g are almost modular forms.

Example: Intersection Numbers

- Kontsevich-Witten: $\big(\mathbb{CP}^1, x(z) = z^2, y(z) = z\big)$.

$$\omega_{g,n}(z_1,\dots,z_n) = 2^{2g-2+n} \sum_{d_1,\dots,d_n} \prod_{i=1}^{n} \frac{(2d_i+1)!!\, dz_i}{z_i^{2d_i+2}} \langle \tau_{d_1}\tau_{d_2}\dots\tau_{d_n}\rangle_g$$

$$\langle \tau_{d_1}\tau_{d_2}\dots\tau_{d_n}\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}.$$

- Weil-Peterson Volumes: $(\mathbb{CP}^1, x(z) = z^2, y(z) = \frac{1}{4\pi}\sin(2\pi z))$.
- Hurwitz Numbers: $(\mathbb{CP}^1, x(z) = \ln(z) - z, y(z) = z)$.

Mirzakhani's relations and ELSV formulae relate these examples to intersection theory.

More Examples

More generally:

Theorem (Eynard)

Given a compact spectral curve $C = (\Sigma, x, y)$ with $g(\Sigma) = 0$ and with simple ramification, there exists a tautological class $\Lambda(C) \in H^*(\overline{\mathcal{M}}_{g,n})$ such that

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{d_1,\ldots,d_n} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(S) \prod_{i=1}^{n} \psi^{d_i}.$$

Can be interpreted as the Givental Group action.

Total ancestor potential of a Semisimple CohFT.

Tau function & Integrable Hierarchies

Theorem (Kontsevich-Witten)

Generating function of ψ-classes intersections is a τ-function of the KdV hierarchy:

$$\exp\left(\sum_{n\geq 0}\sum_{d_1,\dots,d_n}\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g\frac{t_{d_1}\cdots t_{d_n}}{n!}\right).$$

- How do we construct a generating function from $\omega_{g,n}$?
- How do we construct a differential operator that annihilates such a generating function?

Quantum Curves

Quantizing Spectral Curve

Consider the **generating function** of free energies

$$Z = \exp\left(\sum_{g\geq 0} \hbar^{2g-2} F_g\right).$$

The wave function is obtained via a Schlesinger transform

$$\psi_{\mathrm{P}}(z) = \frac{Z\left[ydx \to ydx + \hbar\omega_a^z\right]}{Z\left[ydx\right]} = \exp\left(\frac{1}{\hbar^2}\sum_{k\geq 0} \hbar^k S_k(z)\right).$$

Does there exist a differential operator $\hat{P}(\hat{x}, \hat{y}; \hbar)$ such that

$$\hat{P}(\hat{x}, \hat{y}; \hbar) \cdot \psi_{\mathrm{P}}(z) = 0?$$

Quantum Curves

A **Quantum Curve** $\hat{P}$ of a Spectral Curve C with defining polynomial P is a differential operator in x such that, after normal ordering, it takes the form

$$\hat{P}(\hat{x}, \hat{y}; \hbar) = P(\hat{x}, \hat{y}) + \sum_{n \geq 1} \hbar^n P_n(\hat{x}, \hat{y}),$$

where $\hat{x} = x, \hat{y} = \hbar \frac{d}{dx}$. If $P(x, y)$ is order d in y then P_n are polynomials in $\hat{x}$ and $\hat{y}$ of degree at most $d - 1$ in $\hat{y}$.

A quantum curve is **simple** if the sum is finite.

Genus Zero: Perturbative

Theorem (Bouchard-Eynard)

Given a compact spectral curve $C = (\Sigma, x, y)$ of $g(\Sigma) = 0$, then there exists a simple quantum curve annihilating the wave function:

$$\hat{P}(\hat{x}, \hat{y}; \hbar) \cdot \psi_{\mathrm{P}}(z) = 0.$$

For KdV-r: $(\mathbb{CP}^1, x(z) = z^r, y(z) = z)$, the quantum curve is

$$\left(\hbar^r \frac{d^r}{dx^r} - x\right) \cdot \psi_{\mathrm{P}}(z) = 0.$$

For spectral cureves of higher genus, same methods lead to non-simple quantum curves with non-polynomial expressions for P_n.

Genus One: Non-perturbative

Want a partition function that is modular invariant and background independent.

Non-perturbative partition function:

$$Z_{\mathrm{NP}} = \exp\left(\sum_{g\geq 0} \hbar^{2g-2} F_g\right)$$
$$\times \left(\sum_{r\geq 0} \frac{1}{r!} \sum_{\substack{h_j,d_j\geq 0 \\ 2h_j+d_j-2>0}} \hbar^{\sum 2h_j+d_j-2} \prod_{j=1}^{r} \left(\frac{F_{h_j}^{(d_j)}}{(2\pi i)^{d_j} d_j!}\right) \nabla^{(\sum d_j)} \vartheta_{11}(\zeta_\hbar)\right).$$

Where

$$\zeta_\hbar = \frac{1}{2\pi i \hbar} \int_{\mathcal{B}-\tau\mathcal{A}} y dx, \qquad F_h^{(d)} = \frac{1}{h!} \frac{1}{(2\pi i)^d d!} \oint_{\mathcal{B}} \cdots \oint_{\mathcal{B}} \omega_{h,d}(z_1, \ldots, z_d).$$

Non-Perturbative Wave Function

The corresponding wave function:

$$\psi_{\mathrm{NP}}(z) = \frac{Z_{\mathrm{NP}}\left[ydx \to ydx + \hbar\omega_a^z\right]}{Z_{\mathrm{NP}}\left[ydx\right]}.$$

The wave function $\psi_{\mathrm{NP}}(z)$ satisfies the **quantization condition** if it has an asymptotic expansion as $\hbar \to 0$. In such case we define S_k via:

$$\psi_{\mathrm{NP}}(z) = \exp\left(\frac{1}{\hbar^2}\sum_{k\geq 0}\hbar^k S_k(z)\right).$$

Example

Consider the spectral curve $(\mathbb{C}/\Lambda, x(z) = \wp(z), y(z) = \wp'(z))$ with cycles $\mathcal{A} = [0,1]$ and $\mathcal{B} = [0,\tau]$. It satisfies the Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3.$$

- If $g_2 = 0$ then $\zeta_\hbar = 0$ and it satisfies the quantization condition.
- Quantum curve to $O(\hbar^7)$

$$\hat{P}(\hat{x},\hat{y}) = \hbar^2 \frac{d^2}{dx^2} - 4(x^3 - 1) + \hbar^2 \frac{x}{2^2 3} + \hbar^4 \frac{1}{2^6 3^2} \frac{d}{dx} + \hbar^4 \frac{x^2}{2^8 3^3} + \hbar^6 \frac{x}{2^{12} 3^4} \frac{d}{dx}.$$

- Closed expression not known, simplicity.

Application to Knot invariants

A-polynomial

Let $X = S^3 \setminus \mathbb{K}$ be a knot complement. The **representation variety** $R(\pi_1(X))$ is defined as the quotient

$$R(\pi_1(X)) = \mathrm{Hom}(\pi_1(\mathbb{K}), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C}).$$

Let $\mathfrak{l}, \mathfrak{m}$ be the generators of the boundary torus $\pi_1(\partial X) \cong \mathbb{Z} \times \mathbb{Z}$. Then the **A-polynomial** is the defining polynomial of the algebraic variety in $\mathbb{C}^2$ which is the image of the map

$$\begin{aligned} \chi \colon R(\pi_1(X)) &\to \mathbb{C} \times \mathbb{C} \\ \rho &\mapsto (m, l), \end{aligned}$$

for a representation ρ such that $\rho(\mathfrak{m}) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}$, $\rho(\mathfrak{l}) = \begin{pmatrix} l & 0 \\ t & l^{-1} \end{pmatrix}$.

A-polynomial Spectral Curve

Given $X = S^3 \setminus \mathbb{K}$ with A-polynomial $A_{\mathbb{K}}(m, l) = 0$.

- A-polynomial curve has genus ≥ 1 (except $\mathbf{10}^{(1)}_{152}$ up to 8 crossings).
- It comes with two involutions $m \mapsto 1/m$ and $l \mapsto 1/l$.

Spectral curve is $(C_0, \ln m, \ln l)$ where C_0 is smooth birational to $A_{\mathbb{K}}(m, l) = 0$.

Fact:

- If the involution $\iota\colon (m, l) \mapsto (m, 1/l)$ satisfies $\iota_* = -\mathrm{id}$ then $\zeta_\hbar = 0$ and therefore $(C_0, \ln m, \ln l)$ satisfies the quantization condition.

Jones Polynomial

Let $J_N(\mathbb{K}, q)$ be the N-colored Jones polynomial of a knot $\mathbb{K} \subset S^3$.

Theorem (Garoufalidis-Le)

There exists an operator $\hat{\mathfrak{A}}_{\mathbb{K}} \in \mathbb{Z}[e^{\frac{\hbar}{2}\partial u}, e^u, e^\hbar]$ such that

$$\hat{\mathfrak{A}}_{\mathbb{K}} \cdot J_{u/\hbar}(\mathbb{K}, q = e^{2\hbar}) = 0.$$

AJ Conjecture

In the limit $N \to \infty$, the operator $\hat{\mathfrak{A}}_{\mathbb{K}}$ and the A-polynomial coincide up to a factor.

$\hat{A}$-TR Conjecture

Conjecture (Borot-Eynard)

- The non-perturbative TR wave function on the A-polynomial of a hyperbolic knot in $\mathbb{S}^3$ is annihilated $\hat{\mathfrak{A}}_{\mathbb{K}}$.
- $J_N(\mathbb{K}, q = e^{2\hbar}) \sim C_\hbar B(u) \psi_{NP}(z_u^{(\alpha)})$

Checked up to $O(\hbar^4)$ for the figure eight knot and few other cases.

Open Problems

Open problems

- Quantum Curve Conjecture: Non-perturvative TR wave function is annihilated by a quantum curve.
- $\hat{A}$-TR conjecture: Non-perturbative TR wave function of the A-polynomial of a hyperbolic knot computes the asymptotic expansion of its colored Jones polynomial.

Questions?