FINAL EXAM

Problem 1. Compute the following antiderivatives:

- $\int (e^x + 1)dx = e^x + x + C$

- $\int \left(\frac{3}{x} - \frac{1}{\sqrt{x}}\right) dx = 3\ln(x) - 2\sqrt{x} + C$

- $\int (x^4 - 3x^2 + 10x - 2)dx = \frac{x^5}{5} - x^3 + 5x^2 - 2x + C$

- $\int \cos(x)\sqrt{\sin(x)}dx = \int \sqrt{u}du = \frac{2u^{3/2}}{3} + C = \frac{2\sin^{2/3}(x)}{3} + C.$

  $u = \sin(x) \to du = \cos(x)dx$

- $\int \frac{e^x}{2e^x - 4}dx = \int \frac{1}{2u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2e^x - 4| + C$
  $u = 2e^x - 4 \to du = 2e^x dx \to du/2 = e^x dx.$

Problem 2. Compute the following:

- $\lim_{x\to\infty} \frac{\ln(x)}{x^2-1} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$

  Since ∞/∞, use l'Hopitals Rule.

- $\lim_{x\to 0} \frac{e^{3x}-3x-1}{\cos(4x)-1} = \lim_{x\to 0} \frac{3e^{3x}-3}{-4\sin(4x)} = \lim_{x\to 0} \frac{9e^{3x}}{-16\cos(4x)} = -\frac{9}{16}.$

  Since 0/0, use l'Hopitals Rule (twice).

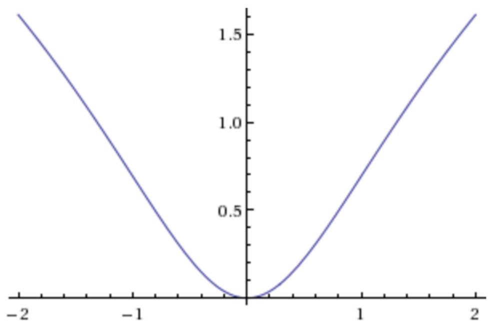
- $\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{3x^2}{1} = 3$

  Since 0/0, use l'Hopitals rule.

- $\int_0^{10} (x^3+x)dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^{10} = \frac{10000}{4} + \frac{100}{2} - (0+0) = 2550.$

- $\int_0^{\pi} (\sin(x)+\cos(x))dx = [-\cos(x)+\sin(x)]_0^{\pi} = (-(-1)+0)-(-1+0) = 2.$

Problem 3. Consider the function $g(x) = \ln(x^2+1)$ in the interval [-2,2] and its graph:



- Without computing their numerical value, write the expressions for the left Riemann sum, right Riemann sum and midpoint Riemann sum for $n = 4$ and sketch them on the graph.

  First we have that $\Delta x = (2 - (-2))/4 = 1$. Then the Riemann Sums are:

  LRS=$\ln(5) + \ln(2) + \ln(1) + \ln(2) = \ln(1) + 2\ln(2) + \ln(5)$.

  RRS=$\ln(2) + \ln(1) + \ln(2) + \ln(5) = \ln(1) + 2\ln(2) + \ln(5)$

  MRS=$\ln(7.25) + \ln(3.25) + \ln(1.25) + \ln(3.25)$

- Compare the sums, what do you realize? Looking at the graph, do you think they are overestimates or underestimates?

  LRS and RRS are the same (because of the symmetry of the function). The sums seem pretty similar, and it is hard to tell if they are overestimates or underestimates.

Problem 4. Consider the function $f(x) = \sqrt[4]{x}$.

- Give the formula for the linear approximation at the point $x = 16$.

  For the linear approximation formula we need to compute $f(16) = 2$, $f'(x) = \frac{1}{4}x^{-3/4}$, and $f'(16) = 1/32$.

- Use this to give the approximate value of $\sqrt[4]{17}$. Using the formula we obtain:

$$L(x) = 2 + \frac{1}{32}(x - 16).$$

  Now we plug in 17 to get

$$\sqrt[4]{17} \simeq L(17) = 2 + \frac{1}{32} = \frac{65}{32}.$$

- Is this an underestimate or an overestimate?

  We compute $f''(x) = \frac{-3}{9\sqrt[4]{x^7}}$, and since $f''(16) < 0$, the function is concave down at that point and it is an overestimate.

Problem 5. Consider the function $f(x) = x^2 - 2x$ and the interval $[-1, 1]$.

- Sketch a graph of the function in this interval.

- Compute the **net area** bounded by the function and the x axis.

- Compute the **real area** bounded by the function and the x axis.

Problem 6. Use the guidelines introduced in class to produce a precise graph of the function

$$f(x) = \frac{x-9}{x+2}.$$

Problem 7. (Spoilers alert!) The First Men, with the aid of the children of the forest and the giants, want to build a huge ice wall to protect Westeros from the White Walkers. The wall has to cover a distance of 100km from Eastwatch-by-the-sea to the Shadow Tower. To do so they have a total amount of 9000km^3 of ice. The effectiveness of the wall against the enemies is given by the following function:

$$E = 1000 - 100t - 10h,$$

where t and h are the thickness and the height of the wall respectively.

- Sketch the situation.

- Find the dimensions of the wall that maximize its effectiveness.

- Some years later, wildlings are attacking!!! Jon Snow is aiming at a giant with his bow from the top of the wall. If the giant is 4km away and approaching the wall with a speed of 20km/h, at what rate does the angle between the arrow's direction and the wall change? Use the dimensions of the wall that you computed above or use your own in case you weren't able to solve it.

Problem 8. Decide true or false. **Justify** your answer.

- The value of the midpoint Riemann Sum is always between the value of the right Riemann Sum and the value of the left Riemann Sum.

- Given a function $f(x)$ and a point a such that $f'(a) = 0$, then $f(a)$ is always either a maximum or minimum value.

- If $f''(x) > 0$, then $f'(x)$ increases.

- The net area and the real area are the same if the function is positive.

- Is the area under the function $f(x) = 1/x^2$ from 1 to ∞ finite or infinite?