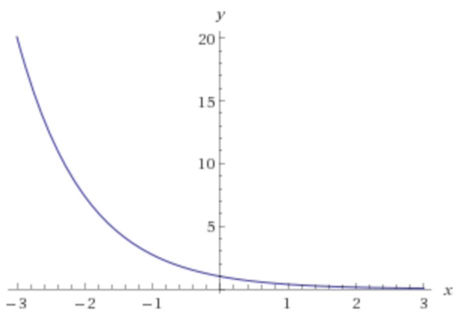
SOLUTIONS

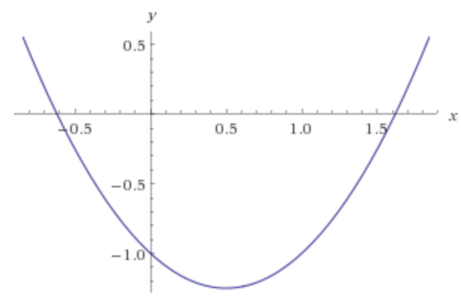
Problem 1. (10 Pts) Each graph below corresponds to a unique function on the list. Label each graph with the letter of its corresponding function.

a. $y = \sin(2x)$ **c.** $y = \ln(x) - 1$ **e.** $y = 2$

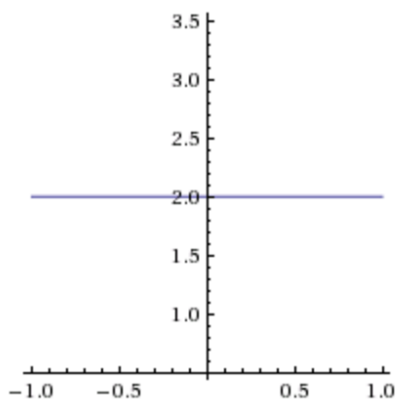
b. $y = e^{-x}$ **d.** $y = x^2 - x - 1$



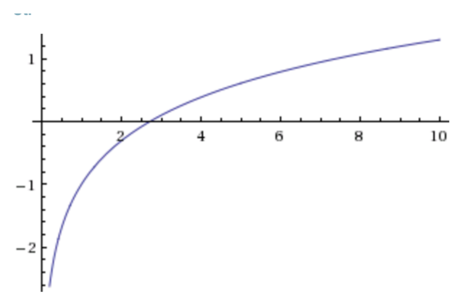
(b.)



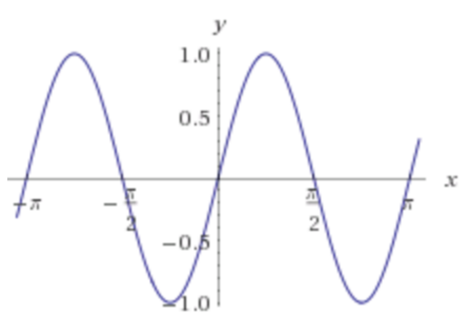
(d.)



(e.)



(c.)



(a.)

Problem 2. (10 pts) Decide if the following situations are possible or impossible. If possible, give an example in form of graph.

- A function continuous at a but not differentiable. (2 pts)

  Possible. A graph with a corner or with a vertical tangent line would work.

- A function differentiable at a but not continuous. (2 pts)

  Impossible. If a function is differentiable at a, then it is continuous at a.

- A function such that $\lim_{x\to a} f(x) \neq f(a)$. (2 pts)

  Possible. Any function not continuous at a.

- A function such that $\lim_{x\to\infty} f(x) = 0$. (2 pts)

  Possible. For example $f(x) = 1/x$, $f(x) = 0$, $f(x) = e^{-x}$ ...

- A function such that $f(2) = 4$ and $f(2) = 5$. (2 pts)

  Impossible. A function can only have one output for each input.

Problem 3. (20 pts) Find the equations of the horizontal asymptotes and the vertical asymptotes, if any, of the following functions:

- $f(x) = \dfrac{2x^2 - 8x}{x - 4}$. (10 pts)

  Horizontal asymptotes:

  $\lim\limits_{x\to\pm\infty} \dfrac{2x^2 - 8x}{x - 4} = \infty$ since degree of numerator is bigger than degree denominator: $2 > 1$.

  There are no horizontal asymptotes.

  Vertical asymptotes:

  First we check when denominator is 0: $x - 4 = 0 \Rightarrow x = 4$. Then we take limit:

  $$\lim_{x\to 4} \frac{2x^2 - 8x}{x - 4} = \lim_{x\to 4} \frac{2x(x-4)}{x - 4} \lim_{x\to 4} 2x = 8$$

  Since 8 is a number there are no vertical asymptotes.

- $g(x) = \dfrac{\sqrt{x^2 + 1}}{2x + 1}$. (10 pts)

  Horizontal asymptotes:

  We take the limit to $+\infty$ dividing by x.

  $$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}}{x}}{\frac{2x+1}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+1}{x^2}}}{2 + \frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{2 + \frac{1}{x}} = \frac{\sqrt{1}}{2} = \frac{1}{2}.$$

  Next, we take the limit to $-\infty$ dividing by $-x$.

  $$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}}{-x}}{\frac{2x+1}{-x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+1}{x^2}}}{-2 - \frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{-2 - \frac{1}{x}} = \frac{\sqrt{1}}{-2} = \frac{-1}{2}.$$

  There are two horizontal asymptotes $\boxed{y = 1/2}$ and $\boxed{y = -1/2}$.

  Vertical asymptotes:

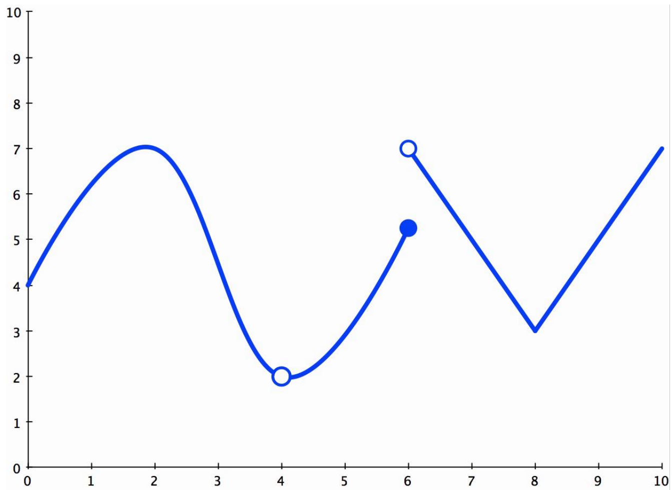
  First we check when denominator is 0: $2x + 1 = 0 \Rightarrow x = -1/2$. Then we take limit:

  $$\lim_{x\to -1/2^-} \frac{\sqrt{x^2+1}}{2x+1} = \frac{\sqrt{5/4}}{0^-} = -\infty$$

$$\lim_{x \to -1/2^+} \frac{\sqrt{x^2+1}}{2x+1} = \frac{\sqrt{5/4}}{0^+} = +\infty,$$

so there is a vertical asymptote $\boxed{x = -1/2}$.

Problem 4. (10 pts) Use the given graph of the function $g(x)$ to answer the following questions.



- At what points is $g(x)$ discontinuous? Why? (4 pts)

  The function is discontinuous at $x = 4$ because $f(4)$ is undefined and at $x = 6$ because the limits from both sides don't agree.

- At what points is $g(x)$ not differentiable? Why? (4 pts)

  The function is not differentiable at $x = 4$ and $x = 6$ because it is not continuous there and at $x = 8$ because there is a corner.

- At what points is the tangent line to the graph of the function horizontal? Sketch it. (2 pts)

  At the point $x = 2$. Note that $x = 4$ is not a correct answer because it is not differentiable there! Since the graph is not precise, if you said near 4 (but not 4) I considered ok.

Problem 5. (20 pts) Consider the equation

$$e^x - 4x = 0,$$

and the interval $(0, 2)$.

- Use the Intermediate Value Theorem to show that there exists a point x_0 in $(0, 2)$ that solves the equation. (Don't give the value of x_0 here). (10 pts)

  Set $f(x) = e^x - 4x$. We have that

  $$f(0) = 1 \text{ and } f(2) = 7.39 - 8 = -0.61.$$

  Since $f(0) > 0 > f(2)$, the IVT assures that there is a number x_0 in $(0, 2)$ such that $f(x_0) = e^{x_0} - 4x_0 = 0$.

- Use the following table of values to find an approximation of x_0 with an error less than 0.15. The values on the table are approximate. (10 pts)

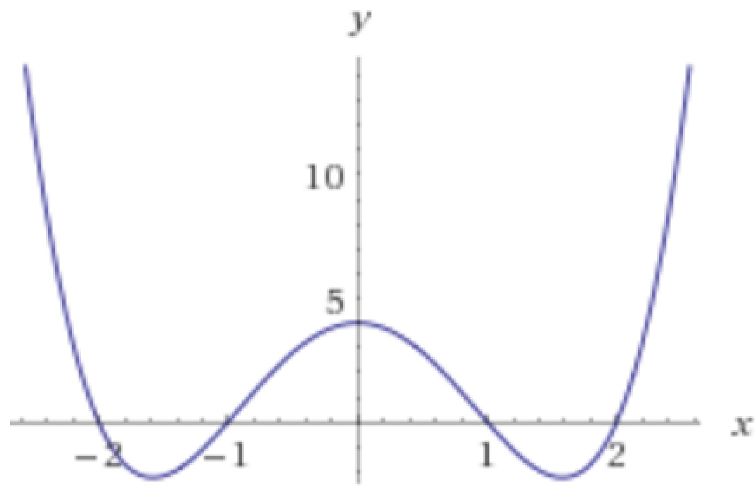
x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
e^x	1	1.28	1.64	2.12	2.72	3.49	4.48	5.75	7.39

  We start off by subdividing the interval $(0, 2)$, the midpoint is $(0 + 2)/2 = 1$. Since $f(1) = e - 4 < 0$ the solution has to be in the interval $(0, 1)$. We repeat the process. We subdivide the interval $(0, 1)$, the midpoint is $= (0 + 1)/2 = 0.5$. Since $f(0.5) = 1.64-2 < 0$ the solution has to be in the interval $(0, 0.5)$. One last time. We subdivide the interval $(0, 0.5)$, the midpoint is $= (0 + 0.5)/2 = 0.25$. Since $f(0.25) = 1.28 - 1 > 0$, the solution is in the interval $(0.25, 0.5)$. The error at this point is $(0.5 - 0.25)/2 = 0.125 < 0.15$, so we are done. We can give the solution as

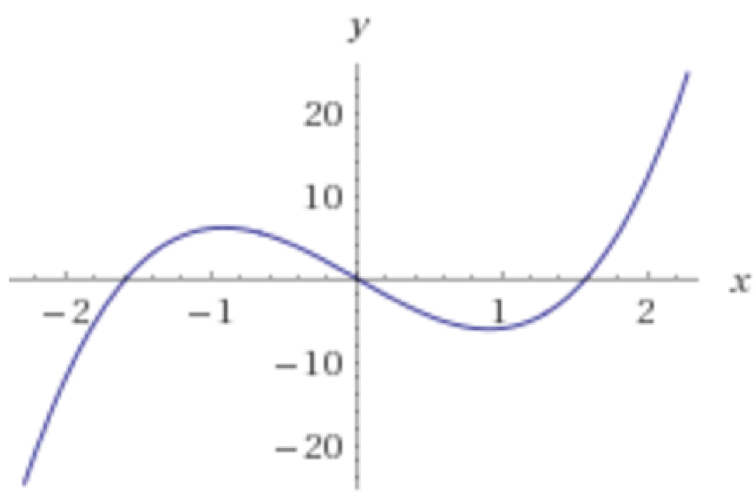
  $$\boxed{x_0 \simeq (0.25 + 0.5)/2 = 0.375}$$

  (the real solution is 0.357403...)

Problem 6. (10 pts) Given the graph of the function $f(x)$, sketch the graph of its derivative $f'(x)$.



To sketch the derivative look at the slope of the original function.



Problem 7. (20 pts) Using the definition of derivative that you prefer, find the tangent line to the graph of

$$f(x) = x^2 + 3$$

at the point (1,4).

Using the definition:

$$m_{\tan} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 4}{h} = \lim_{h\to 0} \frac{h^2 + 2h + 1 + 3 - 4}{h}$$
$$= \lim_{h\to 0} \frac{h^2 + 2h}{h} = \lim_{h\to 0} \frac{h(h+2)}{h} = \lim_{h\to 0} h + 2 = 2.$$

If you used the other definition also ok:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Now we use the formula for the tangent line with slope 2 through the point (1,4):

$$\boxed{y - 4 = 2(x-1)} \text{ or } \boxed{y = 2x + 2}$$