Problem 1. (15 pts) Match each function with its corresponding 3rd-order Taylor polynomial.

- $f(x) = \cos(x)$ (1)
- $g(x) = 1 + \sin(4x)$ (2)
- $h(x) = e^{3x}$ (3)
- $m(x) = \frac{2}{1-x}$ (4)
- $k(x) = 2e^{x}$ (5)

- $p_3(x) = 1 - \frac{x^2}{2!}$ (1)
- $p_3(x) = 2 + 2x + x^2 + \frac{x^3}{3}$ (5)
- $p_3(x) = 2 + 2x + 2x^2 + 2x^3$ (4)
- $p_3(x) = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2}$ (3)
- $p_3(x) = 1 + 4x - \frac{64x^3}{3!}$ (2)

Problem 2. (20 pts) Determine if the following series converge or diverge. State in each case the theorem that applies.

- $\displaystyle\sum_{k=1}^{\infty}(-1)^k\left(\frac{3k^2+3k-1}{4k^3+12}\right)$

  Converges by the alternating series test since

  $$\lim_{k\to\infty}\frac{3k^2+3k-1}{4k^3+12}=0$$

- $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{4^k}\ln(k)$

  Diverges by the ratio test

  $$\lim_{k\to\infty}\frac{(k+1)!\ln(k+1)}{4^{k+1}}\frac{4^k}{k!\ln(k)}=\lim_{k\to\infty}\frac{k+1}{4}=\infty$$

- $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k-\ln(k)}$

  Diverges by the comparison test since $1/k$ diverges and

  $$\frac{1}{k-\ln(k)}>\frac{1}{k}$$

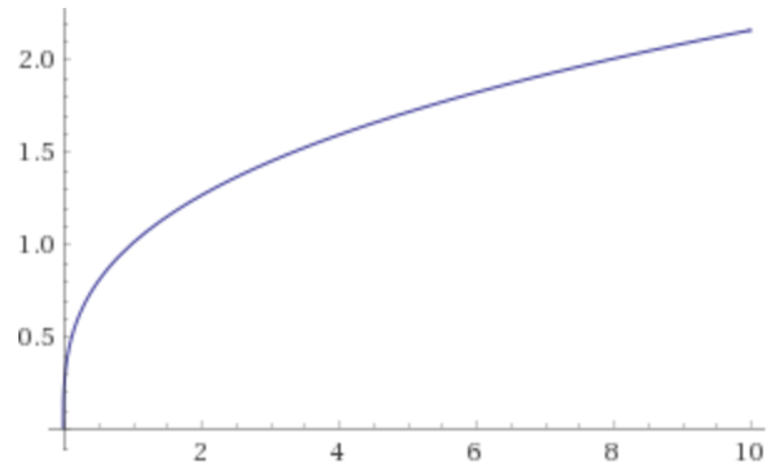
- $\displaystyle\sum_{k=1}^{\infty}\frac{k^k}{2^k k!}$

  Diverges by the ratio test

  $$\lim_{k\to\infty}\frac{(k+1)^{k+1}}{2^{k+1}(k+1)!}\frac{2^k k!}{k^k}=\lim_{k\to\infty}\frac{1}{2}\left(\frac{k+1}{k}\right)^k=\frac{e}{2},$$

  or by the divergence test since k^k has a greater growth rate than both $k!$ and 2^k.

Problem 3. (20 pts) Consider the function $f(x) = \sqrt[3]{x}$



- Compute the 1st and 2nd order Taylor polynomials centred at $a = 8$.

$$p_2(x) = 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}$$

- Use $p_2(x)$ to approximate the value of $\sqrt[3]{8.1}$. (No need to evaluate)

$$\sqrt[3]{8.1} \simeq 2 + \frac{0.1}{12} - \frac{0.01}{288}$$

- Use the remainder formula to estimate the error in the approximation. (No need to evaluate).

$$|R_2(0.1)| < M\frac{0.1^3}{6}$$

were M has to be larger than $f^{(3)}(x) = \frac{10}{27x^{8/3}}$ for x between 0 and 0.1. Since it is decreasing we can take its value at 0

$$M = \frac{10}{27 \cdot 256}$$

Problem 4. (10 pts) Compute the following limits using Taylor series.

- $\lim_{x\to 0} \frac{x - \sin(x)}{x^3} = \lim_{x\to 0} \frac{x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots}{x^3} = \frac{1}{6}$

- $\lim_{x\to 0} \frac{\tan(x) - \arctan(x)}{x^3} = \lim_{x\to 0} \frac{x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots - (x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots)}{x^3} = \frac{2}{3}$

Problem 5. (10 pts) Find the radius of convergence and interval of convergence of the following power series.

- $\displaystyle\sum_{k=0}^{\infty} \frac{(2x-1)^k}{k+1}$

  By the ratio test we have

  $$\lim_{k\to\infty} \frac{|(2x-1)|k}{k+1} = |2x-1| < 1,$$

  therefore $0 < x < 1$. Plugging in $x = 1$ it diverges by limit comparison test with $1/k$ and for $x = 0$ it converges by the alternating series test. Therefore the interval is $[0, 1)$ and the radius $1/2$.

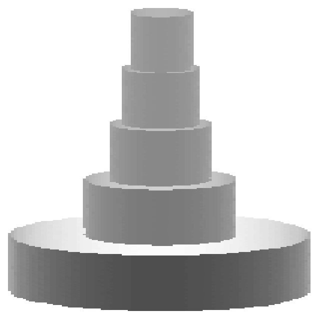
- $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{3^{k-1}}$

  By the ratio test

  $$\lim_{k\to\infty} \frac{|x^{2k+3}|}{3^k} \frac{3^{k-1}}{|x^{2k+1}|} = \frac{|x^2|}{3} < 1$$

  which implies that it converges for $-\sqrt{3} < x < \sqrt{3}$. At the endpoints it doesn't converge by the divergence test. Therefore the interval is $(-\sqrt{3}, \sqrt{3})$ and the radius is $\sqrt{3}$.

Problem 6 (Gabriel's wedding cake)**.** (10pts) To celebrate that the course is over we are going to bake Gabriel's wedding cake. This wedding cake is made by stacking cylindrical cakes of height 1 and with decreasing radiuses $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.



- Write the volume V_n of the nth stacked cake .

$$V_n = 1 \cdot \pi \cdot \left(\frac{1}{n}\right)^2 = \frac{\pi}{n^2}$$

- Find the total volume of the wedding cake.

$$V = \sum \frac{\pi}{n^2} = \pi \sum \frac{1}{n^2} = \frac{\pi^3}{6}$$

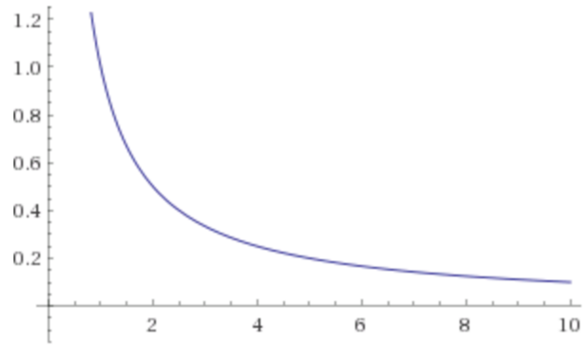
- Write the side area A_n of the nth stacked cake, find the total (side) area and argue why we shouldn't use frosting.

$$A_n = 2\pi \frac{1}{n}$$

$$A = \sum A_n = \sum 2\pi \frac{1}{n} = \infty$$

Because if we use frosting you become broke. You also can't steal it, because you can't steal an infinite amount of frosting

Problem 7 (Gabriel's Horn)**.** (10 pts) To make you didn't forget what we did in the first part of the course: Consider the function $f(x) = 1/x$.



Since this is just the continuous version of the previous problem, you should recover similar answers.

- Show that the area under the function from 1 to ∞ diverges.

$$\int_1^\infty \frac{1}{x}dx = \ln(\infty) - \ln(1) = \infty$$

- Show that the volume of revolution from 1 to ∞ of the function $1/x$ converges.

$$V = \int_1^\infty \pi \frac{1}{x^2}dx = -\frac{\pi}{\infty} + \frac{\pi}{1} = \pi.$$

- Show that the area of revolution, however, diverges.(Hint: Use the general fact that $\sqrt{1+a^2} > 1$).

$$A = \int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}} > \int_1^\infty 2\pi \frac{1}{x}dx = \infty$$

Therefore we can conclude that $A = \infty$.

Problem 8. (5 pts) Write any comments/thoughts you have about the course. Argue why this course may be useful in your future.