Differential Equations and Physics

This exercise is meant to help you go through concepts and calculations introduced in Chapter 7.9 - Introduction to differential equations. Recall that a differential equation is a functional equation: this means that the solution (what we are looking for) is a function $x(t)$ satisfying the given equation. The terms of the equation can include x (unknown function), t (independent variable) and $x', x'', \ldots$ (derivatives). This is an example:

$$x' = 3t + 2\cos(x).$$

Classical mechanics is the area of physics concerned in describing motions of objects. In Newton's formulation, this reduces to solving the differential equation

$$F = mx''(t),$$

where $x''(t)$ is the second derivative of the position $x(t)$, and thus the acceleration. This equation tells us that to describe the motion of an object it suffices to know the force F applied to it.

What is the order of this differential equation? Is it linear?

Consider now a few concrete examples. Throughout this problem assume that $m = 10kg$.

- Gravity. Very close to the earth the gravity can be considered constant. An object of 10kg experiences a force of approximately $F = 100N$.
  - **Write Newton's equation.**
  - **Solve Newton's equation by just using direct integration.**

- **Give the explicit equation if the object starts at rest at $100m$ above the ground.**

- Spring. The force produced by a spring to an object attached to it can be described using Hooke's law
$$F = -kx.$$
  - **Write down Newton's equation in this case.**

  - **Show that $x(t) = A\cos(\sqrt{k/10}t) + B\sin(\sqrt{k/10}t)$ is a solution to the equation.**

  - **Why are there two constants A and B? Do they have any meaning? Find A and B when $x(0) = 1$, $x'(0) = 1/2$ and $k = 10$.**

- Free fall with friction. In a free fall with friction we have both the force of gravity and the friction of the air. If the friction is assumed to be proportional to the velocity (say $10v$) the force is given by
$$F = gravity - friction = 100 - 10v = 100 - 10x'(t).$$
Therefore Newton's equation becomes
$$100 - 10x'(t) = 10x''(t).$$

– **Show that** $x(t) = 10t + Ae^{-t} + B$ **is the general solution to the equation.**

– **Compute the velocity** $v(t) = x'(t)$**.**

– **Compute** $\lim_{t\to\infty} v(t)$ **and discuss the result.**