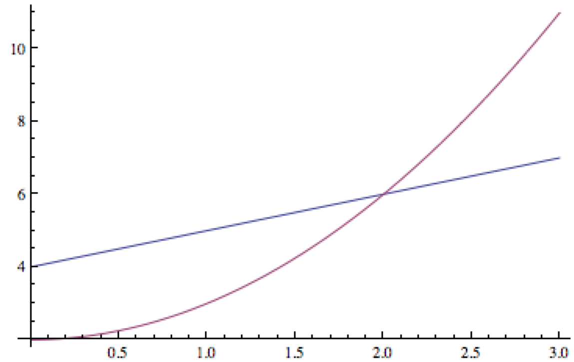
Problem 1. (10 pts) Relate the following concepts:

- Area between functions: $\int_a^b (f(x) - g(x))\, dx$
- Shells method about y: $\int_a^b 2\pi x(f(x) - g(x))dx$
- Lifting a rope: $\int_0^h \rho(y)(h - y)gdy$
- Area of revolution: $\int_a^b 2\pi f(x)\sqrt{(f'(x))^2 + 1}dx$
- Washers method about x: $\int_a^b \pi(f^2(x) - g^2(x))dx$
- Length of a curve: $\int_a^b \sqrt{(f'(x))^2 + 1}dx$

Problem 2. (30 pts) Consider the region R bounded by the curves $y = x + 4, x = 0$, and $y = x^2 + 2$.



- Compute the area of the region.

$$A = \int_0^2 (x + 4 - (x^2 + 2))dx = [-x^3/3 + x^2/2 + 2x]_0^2 = 10/3$$

- Write and evaluate a single integral that gives the volume of the solid generated when R is revolved about the x-axis.

  Using Washers

$$V_x = \int_0^2 \pi((x + 4)^2 - (x^2 + 2)^2)dx = \pi \int_0^2 (-x^4 - 3x^2 + 8x + 12)dx =$$

$$\pi[-x^5/5 - x^3 + 4x^2 + 12x]_0^2 = \pi 128/5.$$

- Write and evaluate a single integral that gives the volume of the solid generated when R is revolved about the y-axis.

  Using Shells

$$V_y = \int_0^2 2\pi x(x+4-(x^2+2))dx = 2\pi \int_0^2 (-x^3+x^2+2x)dx = [-x^4/4+x^3/3+x^2]_0^2 = \pi 16/3$$

- Suppose that S is a solid whose base is R and whose cross sections perpendicular the x-axis are squares. Write and evaluate a single integral that gives the volume of S.

  Using slicing method

$$V_S = \int_0^2 (x + 4 - (x^2 + 2))^2 dx = \int_0^2 (x^4 - 2x^3 - 3x^2 + 4x + 4)dx =$$

$$= [x^5/5 - x^4/2 - x^3 + 2x^2 + 4x]_0^2 = 32/5$$

Problem 3. (15 pts) Find the length of the following curves:

- $y = \sqrt{(x-1)^3}$ in the interval $[6, 10]$.

  First $y' = \dfrac{3}{2}\sqrt{x-1}$. Then $y'^2 = \dfrac{9}{4}(x-1)$ and

  $$L = \int_6^{10} \sqrt{9/4(x-1)+1}dx = \int_6^{10} \sqrt{9x/4-5/4}dx = [8/27(9x/4-5/4)^{3/2}]_6^{10} =$$

  $$\frac{85\sqrt{85}-343}{27} = 16.321.$$

- $y = \dfrac{(x^2+2)^{3/2}}{3}$ in the interval $[1, 2]$. (Hint: The integral becomes easy if you simplify enough).

  Again $y' = x\sqrt{x^2+2}$. Then $y'^2 = x^2(x^2+2) = x^4+2x^2$ and

  $$L = \int_1^2 \sqrt{x^4+2x^2+1}dx = \int_1^2 (x^2+1) = [x^3/3+x]_1^2 = 10/3$$

Problem 4. (15 pts) A biologist is researching a newly-discovered species of bacteria. At time $t = 0$ hours, she puts 100 bacteria into what she has determined to be a favourable growth medium. Six hours later, she measures 300 bacteria. No need to evaluate the expressions.

- Assuming exponential growth, what is the growth constant k for the bacteria?

  Consider a model $B(t) = B_0 e^{kt}$ where B stands for the number of bacteria. Since at $t = 0$ there are 100 bacteria we have $B_0 = 100$. Thus

  $$300 = B(6) = 100e^{k6} \Rightarrow k = \ln(3)/6.$$

- How many bacteria will she measure after 10 hours?

  Computing

  $$B(10) = 100e^{\ln(3)/6 \cdot 10} = 100e^{5\ln(3)/3}$$

- How long will she have to wait to obtain 400 bacteria?

  Solving for t

  $$400 = 100e^{t\ln(3)/6} \;\Rightarrow t = \frac{6\ln 4}{\ln 3}$$

Problem 5. Compute the following integrals/antiderivatives:

- $\int (x^3 + \sqrt{x})dx = \frac{x^4}{4} + \frac{2x^{3/2}}{3} + C$

- $\int 3\sin(x)e^{\cos(x)}dx = -3e^{\cos(x)} + C$ (Using $u = \cos(x)$.)

- $\int_0^3 (x^2 - 3x + 1)dx = [\frac{x^3}{3} - \frac{3x^2}{2} + x]_0^3 = -3/2$

- $\int \frac{2x+1}{x^2+x+10}dx = \ln(x^2 + x + 10) + C$ (Using $u = x^2 + x + 10$).

Problem 6. A climber made it to the top of the pitch and now needs to pull up the hanging rope. The rope extends from the top of the climb at 100m all the way to the floor. Assume that, because of gravity, the rope is thicker in the lower end and its density profile in kg/m is given by $\rho(y) = 5 - 0.04y$ for $0 \leq y \leq 100$.

- Find the mass of the rope.

$$m = \int_0^{100} (5 - 0.04y)dy = [5y - 0.02y^2]_0^{100} = 500 - 200 = 300kg.$$

- Find the work done by the climber to pull the rope all the way up. Take $g = 10m/s^2$.

$$W = \int_0^{100} (100 - y)10(5 - 0.04y)dy = 10 \int_0^{100} (500 - 5y - 4y + 0.04y^2)dy =$$

$$10 \left[500y - \frac{9y^2}{2} + \frac{0.04y^3}{3}\right]_0^{100} = 10(50000 - 90000/2 + 40000/3) = 550000/3$$

- How many energy bars should she/he eat to compensate the work done if each bar provides 10J?

  Many (or 55000/3).