Problem 1. (15 pts) Compute the following integrals.

- $\displaystyle\int_0^{\pi/4} x\sin(2x)\,dx.$

  Integrate by parts with $u = x, dv = \sin(2x)$. Then $du = 1, v = -\frac{\cos(2x)}{2}$.

$$\left[-\frac{x\cos(2x)}{2}\right]_0^{\pi/4} + \int_0^{\pi/4} \frac{\cos(2x)}{2} = -\frac{\pi\cos\left(\frac{\pi}{2}\right)}{8} + 0 + \left[\frac{\sin(2x)}{4}\right]_0^{\pi/4} = \frac{1}{4}.$$

- $\displaystyle\int_0^{\infty} xe^{-2x^2}\,dx$

  Direct improper integral. You can use u-substitution $u = x^2$.

$$\lim_{b\to\infty}\left[-\frac{e^{-2x^2}}{4}\right]_0^b = -\lim_{b\to\infty} e^{-2b^2} + \frac{1}{4} = \frac{1}{4}$$

- $\displaystyle\int_{-\infty}^{0} \frac{3dx}{1+x^2}$

  Arrange and direct integration. Trigonometric substitution also valid.

$$\int_{-\infty}^{0} \frac{3dx}{(1+x^2)} = \lim_{a\to-\infty}\left[3\arctan(x)\right]_a^0 = \lim_{a\to-\infty} -3\arctan(a) = \frac{3\pi}{2}$$

Problem 2. (10 pts) Compute the following antiderivatives. State the method used.

- $\displaystyle\int \frac{3dx}{x^3+5x^2+6x} = \int \frac{3dx}{x(x+2)(x+2)} = \int \left(\frac{1}{2x} - \frac{3}{2(x+2)} + \frac{1}{x+3}\right)$
$\displaystyle= \frac{\ln|x|}{2} - \frac{3\ln|x+2|}{2} + \ln|x+3| + C$

- $\displaystyle\int x^2 \sin(x)\, dx = -x^2\cos(x) - \int 2x\cos(x) = -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)dx$
$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$

Problem 3. (20 pts) Write either and explicit or recursive formula for the first two sequences. Find the first 5 terms for the rest.

- $\{1, 4, 9, 16, 25, 36, \ldots\}$. The general term is $a_n = n^2$.
- $\{\frac{-1}{3}, \frac{1}{5}, \frac{-1}{7}, \frac{1}{9} \ldots\}$. The general term is $(-1)^n/(2n+1)$.
- $a_n = n\cos(n\pi/2)$. The first terms are $\{0, -2, 0, 4, 0, -6, 0, 8, \ldots\}$.
- $a_{n+1} = 2a_n + 2$, $a_1 = 0$. The first terms are $\{0, 2, 6, 14, 30, 62, \ldots\}$.
- $a_n = \frac{(-2)^n}{n^2+1}$. The first terms are $\{-1, 4/5, -4/5, 16/17, -32/26, \ldots\}$.

Problem 4. (10 pts) Use trigonometric substitution to evaluate the following integral.

$$\int \frac{dx}{x^2\sqrt{9+x^2}}.$$

This corresponds to the substitution $x = 3\tan(\theta)$. Then $dx = 3\sec^2(\theta)d\theta$. Thus the integral becomes:

$$\int \frac{3\sec^2(\theta)}{9\tan^2(\theta)3\sec(\theta)}d\theta = \int \frac{\cos(\theta)}{9\sin^2(\theta)}d\theta$$

Now use u-substitution $u = \sin(x)$, $du = \cos(x)$ to get

$$\int \frac{du}{9u} = \frac{-1}{9u} + C = \frac{-1}{9\sin(\theta)} + C$$

Use triangle picture to find $\sin(\theta)$ in terms of x.

$$\tan(\theta) = x/3 \Rightarrow \sin(\theta) = \frac{x}{\sqrt{x^2+9}}$$

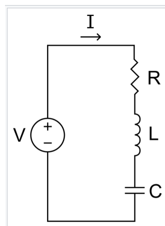
So the result is

$$-\frac{\sqrt{x^2+9}}{9x} + C$$

Problem 5. (30 pts) An RLC circuit is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel. The differential equation that describes the behaviour of the intensity $I(t)$ as a function of time is

$$I''(t) + 2\alpha I'(t) + \omega_0^2 I(t) = 0,$$

where α and ω are constant parameters that depend on R, L and C.



Suppose that the equations of two different RLC circuits (A) and (B) are given:

(A) $I''(t) + 10I'(t) + 16I(t) = 0$

(B) $I''(t) + 8I'(t) + 25I(t) = 0$

- Show that the following function is the general solution to the differential equation for the circuit (A):
$$I(t) = Ae^{-2t} + Be^{-8t}.$$
Compute first $I'(t) = -2Ae^{-2t} - 8Be^{-8t}$ and $I''(t) = 4Ae^{-2t} + 64Be^{-8t}$. Then plug in original equation
$$(4 - 20 + 16)Ae^{-2t} + (64 - 80 + 16)Be^{-8t} = 0.$$

- Find A an B in the case where $I(0) = 10$ and $I'(0) = -50$.
First condition gives $10 = I(0) = A+B$. Second condition gives $-2A-8B = -50$ and thus $2A + 2B = 50$. Therefore $A = B = 5$.

- Show that the following function is a particular solution to the differential equation for the circuit (B):

  $$I(t) = e^{-4t}\cos(3t)$$

  Same a before. Find $I'(t) = -4e^{-4t}\cos(3t) - 3e^{-4t}\sin(3t)$ and $I''(t) = 7e^{-4t}\cos(3t) + 24e^{-4t}\sin(3t)$. Then

  $$(7 - 32 + 25)e^{-4t}\cos(3t) + (24 - 24)e^{-4t}\sin(3t) = 0.$$

- Compare and discuss both solutions. What happens with the intensity $I(t)$ in each case?

  In (A) the intensity doesn't oscillate, so in particular it is always in the same direction. In (B), the intensity oscillates, meaning that the electric charges will change direction. In both cases the amplitude decreases.

Problem 6. (15 pts) True or false (BONUS: Give examples or counterexamples).

If $\{a_n\}_{n=1}^{\infty}$ is a sequence, $S = \sum_{n=1}^{\infty} a_n$ the sum of its terms (infinite series) and $L = \lim_{n\to\infty} a_n$ its limit as n tends to ∞, then:

- The limit L always exists. False, consider $\{1, -1, 1, -1, 1, -1, \ldots\}$

- The sum S always converges. False, consider $\sum_{n=0}^{\infty} 1$.

- If $L = 0$, then S always converges. False, consider harmonic sequence $1/n$.

- If S converges, then $L = 0$. True, divergence test theorem.

- If $L = 3$ then S diverges. True, divergence test theorem.