FINAL EXAM

Last name: Name:

BUID:

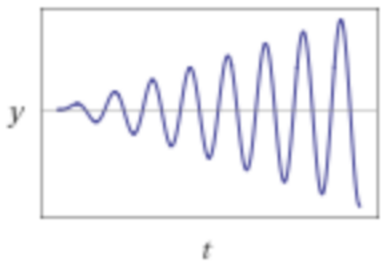
Please do all of your work in this exam booklet and make sure that you cross any work that we should ignore when we grade. Books and extra papers are not permitted. If you have a question about a problem, pease ask. Remember: answers that are written logically and clearly will receive higher scores. Calculators are not allowed.

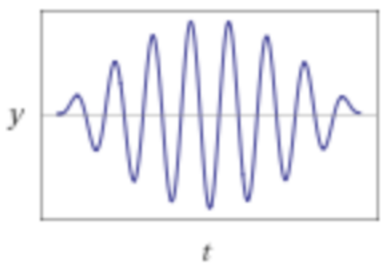
Problem n^o^	**Possible points**	**Score**
1	12	
2	10	
3	18	
4	15	
5	25	
6	15	
7	5	
Total:	100	

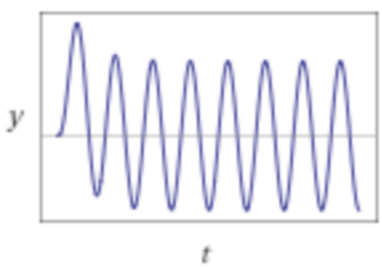
Problem 1. Consider the general equation for a harmonic oscillator with sinusoidal forcing:

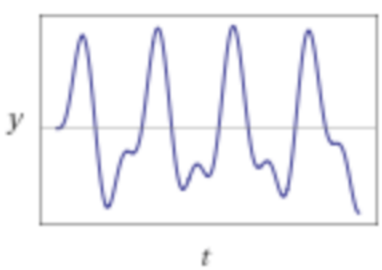
$$y'' + py' + qy = \sin(\omega t).$$

Each of the following graphs correspond to a forced harmonic oscillator with equation above. Indicate if the oscillator is damped or undamped, if it exhibits beating, resonance, etc. and choose possible values of p, q and ω in each case.









Problem 2. Match each real life event in the left column with the mathematical model in the right column that best approximates it.

- Car brakes.
- Spinner.
- Instantaneous impulse.
- Yo-yo.
- The Moon orbiting the Earth.

- Oscillator with sinusoidal forcing.
- Underdamped oscillator.
- Undamped oscillator.
- Overdamped oscillator.
- Delta function.

Problem 3. TRUE or FALSE. You **DON'T** need to justify your answers.

- Nonlinear systems can have periodic solutions.
- The amplitude of the forced response depends on the forcing frequency.
- Damped oscillators can exhibit resonance.
- Linearization can fail if the linearized system is a saddle.
- Linearization can fail if the linearized system is a center.
- The error of Euler's method depends on the size of the time step Δt.

Problem 4. An undamped harmonic oscillator with natural angular frequency $\omega = 2$ is placed on a table. Suddenly, at $t = 3$, the table is tilted. Which of the following differential equations correctly describes this situation?

- $y'' + 3y' + y = 10$
- $y'' + 4y = u_3(t)$
- $y'' + 4y = \sin(t)$
- $y'' + 10y = u_3(t)$

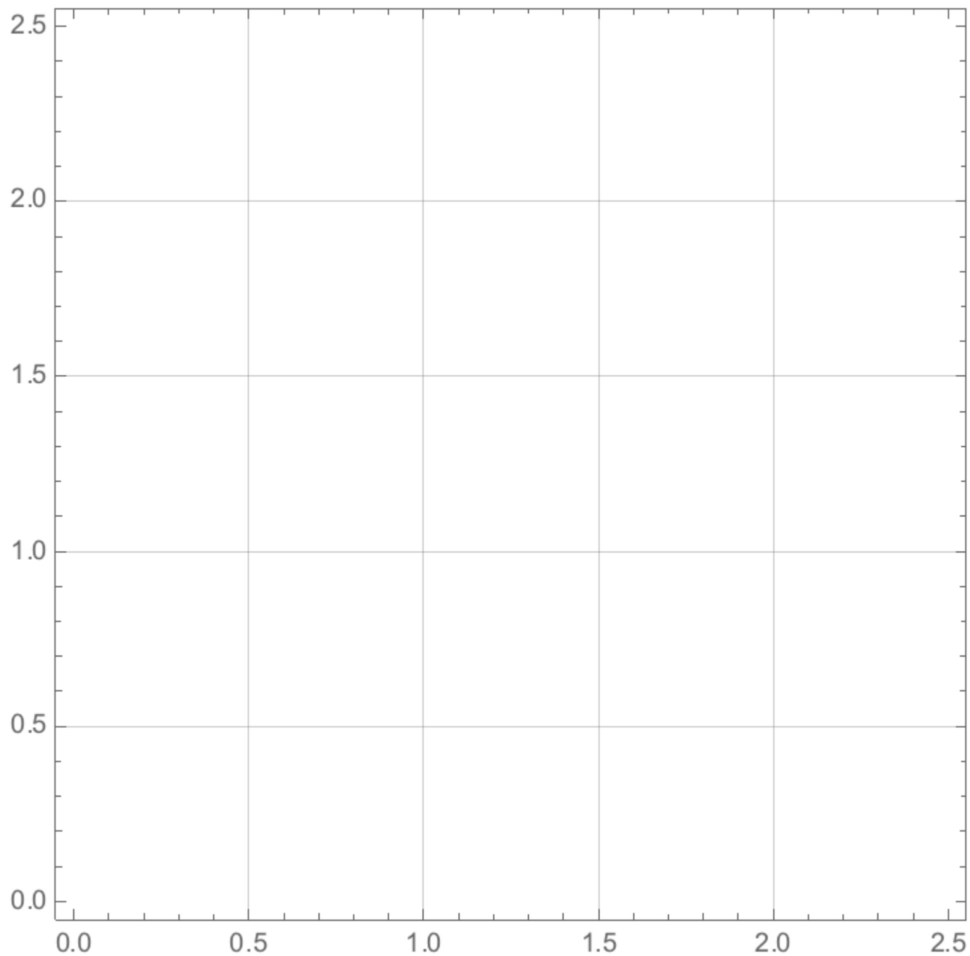
For the chosen equation, solve the initial value problem with initial conditions $y(0) = 0$ and $y'(0) = 1$,

Problem 5. A good model for interacting species is given by the Volterra-Lotka equations. Consider two species x and y whose interaction is modelled by the following nonlinear system of differential equations:

$$\begin{cases} \frac{dx}{dt} = x(2 - x - y) = 2x - x^2 - xy \\ \frac{dy}{dt} = y(1 - 2x) = y - 2xy \end{cases}$$

Since x and y must be positive, consider only the first quadrant.

- Describe the type of interaction. For the first equation, what does the x^2 term account for?
- Find the equations of the x and y nullclines and sketch them (use the graph paper provided).
- Determine the direction of the vector field on the nullclines.
- Find the equilibrium points and determine the type of equilibrium at each of them.
- Identify two regions from where solutions can't escape.
- Describe the long term behaviour of solutions with initial condition $(x_0, y_0) = (1, 1/2)$ and $(1/4, 1)$. Sketch these solutions on the graph.


2.5
2.0
1.5
1.0
0.5
0.0
0.0 0.5 1.0 1.5 2.0 2.5

Problem 6. Solve the following initial value problem

$$\frac{d^2y}{dt^2} = 2\delta_4(t), \qquad y(0) = 0, \qquad y'(0) = -1.$$

(HINT: What is the inverse Laplace transform of $1/s^2$?)

What could this differential equation be describing?

- A soccer ball being kicked against a wall.
- A kid riding a bike.
- Sounds waves interfering.
- A revolving door.

Problem 7. Explain why differential equations may be helpful for you in the future. Do you prefer numeric, analytic or qualitative methods? Why?

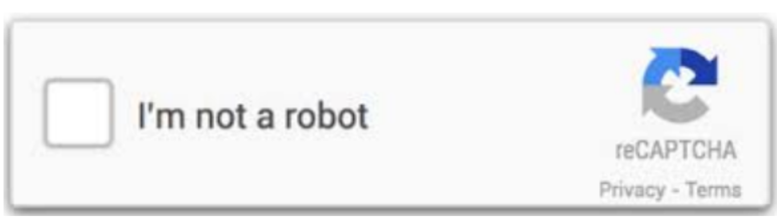


Table 6.1
Frequently Encountered Laplace Transforms.

$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$	$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a} \quad (s > a)$	$y(t) = t^n$	$Y(s) = \dfrac{n!}{s^{n+1}} \quad (s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \dfrac{\omega}{s^2 + \omega^2}$	$y(t) = \cos \omega t$	$Y(s) = \dfrac{s}{s^2 + \omega^2}$
$y(t) = e^{at} \sin \omega t$	$Y(s) = \dfrac{\omega}{(s-a)^2 + \omega^2}$	$y(t) = e^{at} \cos \omega t$	$Y(s) = \dfrac{s-a}{(s-a)^2 + \omega^2}$
$y(t) = t \sin \omega t$	$Y(s) = \dfrac{2\omega s}{(s^2 + \omega^2)^2}$	$y(t) = t \cos \omega t$	$Y(s) = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s} \quad (s > 0)$	$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$

Table 6.2
Rules for Laplace Transforms:
Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants α and a.

Rule for Laplace Transform	Rule for Inverse Laplace Transform
$\mathcal{L}\left[\dfrac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) = sY(s) - y(0)$	
$\mathcal{L}[y + w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$	$\mathcal{L}^{-1}[Y + W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$
$\mathcal{L}[\alpha y] = \alpha\mathcal{L}[y] = \alpha Y(s)$	$\mathcal{L}^{-1}[\alpha Y] = \alpha\mathcal{L}^{-1}[Y] = \alpha y(t)$
$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y] = e^{-as}Y(s)$	$\mathcal{L}^{-1}[e^{-as}Y] = u_a(t)y(t-a)$
$\mathcal{L}[e^{at}y(t)] = Y(s-a)$	$\mathcal{L}^{-1}[Y(s-a)] = e^{at}\mathcal{L}^{-1}[Y] = e^{at}y(t)$