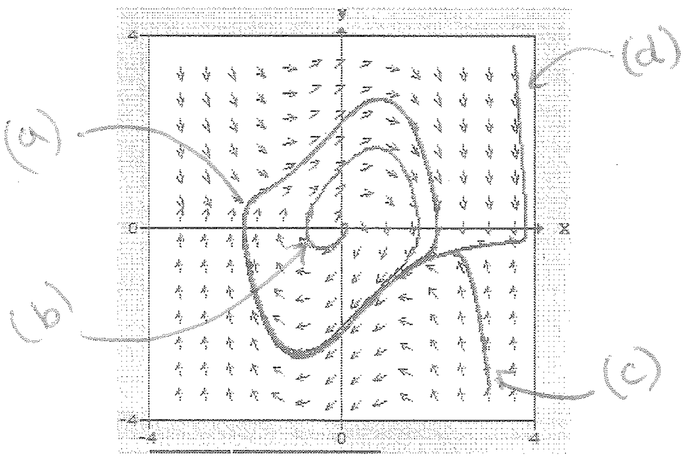
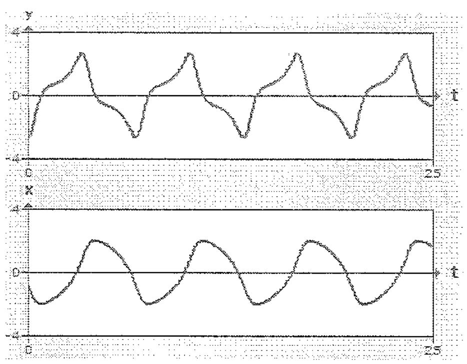
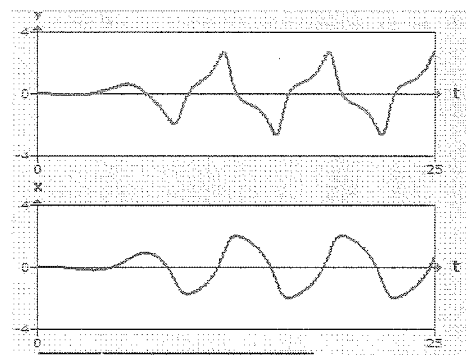
Problem 1. Consider the four solution curves in the phase portrait, and the four pairs of $x(t)$ and $y(t)$ graphs shown below.



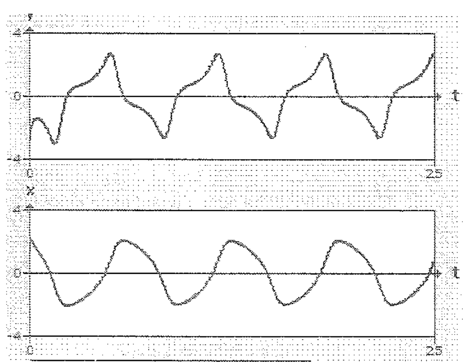
Match each solution curve with its corresponding graph. Beware that in the long term all solutions tend to the same behaviour!



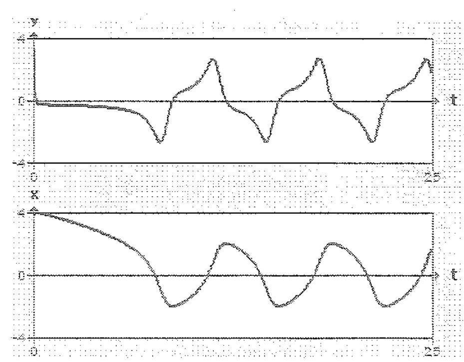
(a)



(b)



(c)



(d)

Problem 2. Consider the non-linear system

$$\begin{cases} \frac{dx}{dt} = x - y \\ \frac{dy}{dt} = y - x^2 \end{cases}$$

Starting at $Y(0) = (2, 2)$. Do Euler's method up to $t = 2$ with time step $\Delta t = 1$.

K	t_k	X_k	Y_k	$f_k(x_k, y_k)$	$g_k(x_k, y_k)$
0	0	2	2	0	-2
1	1	2	0	2	-4
2	2	4	-4		

Repeat the process with $\Delta t = 0.5$ up to $t = 1.5$. Compare the results, what do you notice?

k	t_k	x_k	y_k	$f_k(x_k, y_k)$	$g_k(x_k, y_k)$
0	0	2	2	0	-2
1	0.5	2	1	1	-3
2	1	2.5	-0.5	3	
3	1.5	4	-3.875		6.75

They both are at a similar point, but the first one is there at $t=2$, and the second at $t=1.5$!

Problem 3. Consider the linear system of differential equations

$$\frac{dY}{dt} = \begin{pmatrix} 2 & 0 \\ 5 & -2 \end{pmatrix}.$$

- Find the eigenvalues and eigenvectors associated to this system.
- Find the general solution.
- Sketch the phase portrait for this system and indicate the type of equilibrium.
- Find the solution satisfying $Y(0) = (1, 1)$.

- $\det\begin{pmatrix} 2-\lambda & 0 \\ 5 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) = 0 \Rightarrow \begin{matrix}\lambda = 2 \\ \lambda = -2\end{matrix}$

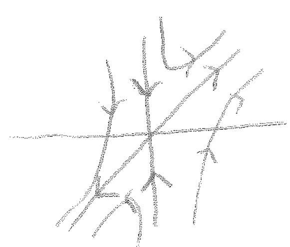
$\underline{\lambda = 2}$

$\begin{pmatrix} 0 & 0 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 5x = 4y \Rightarrow v_1 = (4,5)$

$\underline{\lambda = -2}$

$\begin{pmatrix} 4 & 0 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4x = 0 \Rightarrow v_2 = (0,1)$

- $Y(t) = k_1 e^{2t}\begin{pmatrix} 4 \\ 5 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- 

SADDLE

- $Y(0) = k_1\begin{pmatrix} 4 \\ 5 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} 4k_1 = 1 \\ 5k_1 + k_2 = 1 \end{matrix} \Rightarrow \begin{matrix} k_1 = 1/4 \\ k_2 = 1 - \frac{5}{4} = -\frac{1}{4} \end{matrix}$

$Y(t) = \frac{1}{4}e^{2t}\begin{pmatrix} 4 \\ 5 \end{pmatrix} - \frac{1}{4}e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

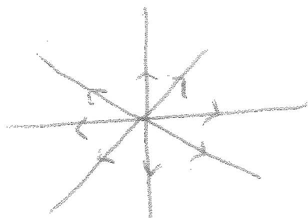
Problem 4. Short answer questions.

- Write the equation of an underdamped harmonic oscillator. Choose values for m, b and k.

want $b^2-4mk<0$, so for example $b=m=k=1$

$y''+y'+y=0$

- Sketch the phase portrait of the system $\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$.



all lines are straight-line solutions.

- Convert the second order equation $3y''+2y'-7y=0$ into a system.

$$\begin{cases} y' = v \\ v' = \dfrac{-2y'+7y}{3} = -\dfrac{2}{3}v + \dfrac{7}{3}y \end{cases}$$

- Find the tangent vector to the solution curve at the point $(x, y) = (2, 1)$ of the system

$$\begin{cases} \frac{dx}{dt} = yx - e^y \\ \frac{dy}{dt} = \frac{y-3}{x^2+1} \end{cases}$$

$$\begin{pmatrix} 1\cdot 2 - e^1 \\ \frac{1-3}{2^2+1} \end{pmatrix} = \begin{pmatrix} 2 - e^1 \\ -\frac{2}{5} \end{pmatrix}$$

- TRUE or FALSE: Solution curves of a non-autonomous system can intersect in the phase plane. Justify your answer in virtue of uniqueness and existence theorem.

True.

Problem 5. Consider the system of differential equations

$$\frac{dY}{dt} = \begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}.$$

- Find the eigenvalues associated to this system.
- Find the general solution of this system.
- Sketch the phase plane for this system, indicate the type of equilibrium and the period of the oscillations.

- $\det\begin{pmatrix} -\lambda & -2 \\ 8 & -\lambda \end{pmatrix} = \lambda^2+16=0 \Rightarrow \lambda = \pm 4i$

- $\begin{pmatrix} -4i & -2 \\ 8 & -4i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad 8x-4iy=0 \Rightarrow 2x-iy=0$

  $v=(i,2)$

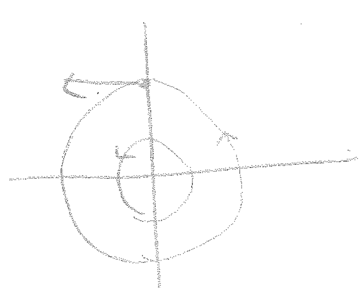
- $Y_c(t) = e^{4it}\begin{pmatrix} i \\ 2 \end{pmatrix} = (\cos(4t)+i\sin(4t))\begin{pmatrix} i \\ 2 \end{pmatrix}$

  $= \begin{pmatrix} i\cos(4t) - \sin(4t) \\ 2\cos(4t)+2i\sin(4t) \end{pmatrix} = \begin{pmatrix} -\sin(4t) \\ 2\cos(4t) \end{pmatrix} + i\begin{pmatrix} \cos(4t) \\ 2\sin(4t) \end{pmatrix}$

  $Y(t) = k_1\begin{pmatrix} -\sin(4t) \\ 2\cos(4t) \end{pmatrix} + k_2\begin{pmatrix} \cos(4t) \\ 2\sin(4t) \end{pmatrix}$

- Test $(0,1)$, $\left.\frac{dY}{dt}\right|_{(0,1)} = \begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$P = \frac{2\pi}{4} = \frac{\pi}{2}$



Problem 6. Bob and Paul decide to open two small cafés near the BU campus. Having two beverage shops close will make the area more popular to students. However, both shops will compete for their costumers. If $x(t)$ is the profit of Paul's café at time t, and let $y(t)$ is the profit of Bob's café, we can write the system of linear equations

$$\frac{dY}{dt} = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix} Y.$$

- From a real-life perspective, comment on the fact that the top-right element of the matrix is positive, but the bottom-left is negative.
- Find the eigenvalue of this system.
- Write down the general solution.
- Regardless of the starting point, what will happen in the long-term? Is that good for Bob and Paul?

∘

∘ $\det\begin{pmatrix} -5-\lambda & 1 \\ -1 & -3-\lambda \end{pmatrix} = \lambda^2+8\lambda+15+1 = (\lambda+4)^2 = 0$

$\lambda = -4$ repeated.

∘ Take a general vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, and compute $(A+4I)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

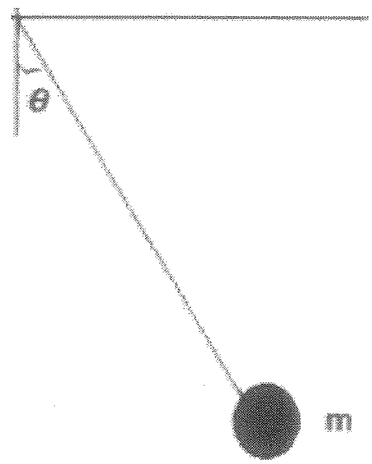
$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}$$

So $Y(t) = e^{-4t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-4t}\begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}$

Recall general solution for repeated eigenvalue $Y(t) = e^{\lambda t}V_0 + te^{\lambda t}V_1$

∘ $Y(t) \to (0,0)$, so no profit

Problem 7. A simple pendulum can be modelled via a harmonic oscillator of mass $m = 1$ and spring constant $k = g/l$, where $g = 10$ is the gravity and $l = 2$ is the length of the pendulum.



Suppose that the pendulum swings without friction $b = 0$. A tiny device is then placed at the end of the rope, absorbing the energy of the swing and thus introducing some damping $b > 0$.

- Write down the corresponding second order equation.

Since $m=1$
$b>0$
$k = \frac{10}{2} = 5$

$y'' + by' + 5y = 0$

- Find the values of b for which the pendulum will not oscillate.

overdamped $b^2 > 4mk \Rightarrow b^2 > 20$ $b > 2\sqrt{5}$

- Find the values of b for which the pendulum will oscillate for all t but the amplitude of the oscillations will decrease to 0. What is the period of the oscillations?

underdamped $b^2 < 4mk \Rightarrow b < 2\sqrt{5}$. Period is $\frac{2\pi \cdot 2m}{\sqrt{4mk - b^2}} = \frac{4m\pi}{\sqrt{4mk-b^2}} = \frac{4\pi}{\sqrt{20-b^2}}$

- Find the value of b for which the pendulum will not oscillate but will swing faster.

critically damped $b^2 = 4mk$ $b = 2\sqrt{5}$