FINAL EXAM

Last name: Name:

BUID:

Please do all of your work in this exam booklet and make sure that you cross any work that we should ignore when we grade. Books and extra papers are not permitted. If you have a question about a problem, pease ask. Remember: answers that are written logically and clearly will receive higher scores. Calculators are not allowed.

Problem n^o	**Possible points**	**Score**
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Problem 1. Each of the graphs below corresponds to a forced harmonic oscillator with equation

$$y'' + py' + qy = \sin(\omega t),$$

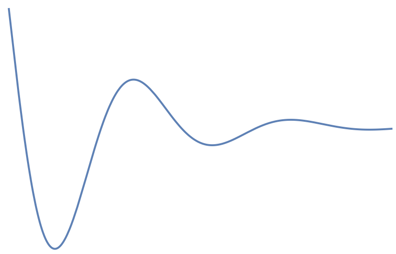
for some specific values of p, q and ω. Choose, from the list below, the type of oscillator that corresponds to each of the graphs **and** indicate possible values of p, q and ω.

(i) Over-damped with sinusoidal forcing.

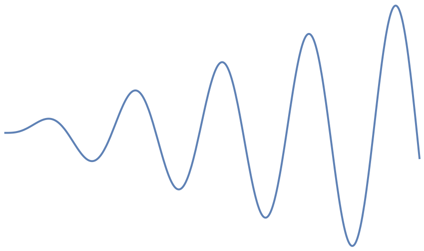
(ii) Undamped with sinusoidal forcing. Resonant.

(iii) Under-damped with no forcing.

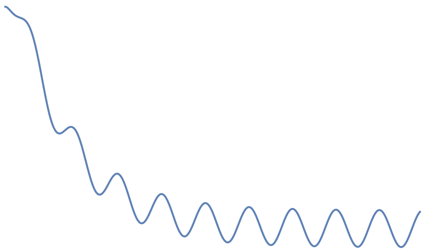
(iv) Undamped with sinusoidal forcing. Beating.



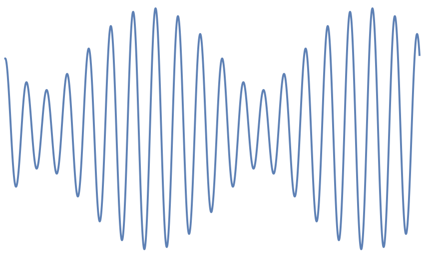
Type: $p =$ $q =$ $\omega =$



Type: $p =$ $q =$ $\omega =$



Type: $p =$ $q =$ $\omega =$



Type: $p =$ $q =$ $\omega =$

Problem 2. Answer the following questions regarding Laplace transforms. (Hint: completing the square may be useful)

- Compute $\mathcal{L}^{-1}\left[\frac{s+1}{s^2-2s+6}\right]$.

- Compute $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2-2s+6}\right]$.

- Solve the following initial value problem using Laplace transform method. The previous computations should be of help.

$$y'' - 2y' + 6y = \delta_2(t), \qquad y(0) = 1, \qquad y'(0) = 3.$$

Problem 3. Short questions.

(i) The amplitude of the forced response of a damped harmonic oscillator with sinusoidal forcing is
$$A(p, q, \omega) = \frac{1}{\sqrt{(q - w^2)^2 + p^2\omega^2}}.$$
If $p = 1$ and $\omega = 6$, what value of q makes the amplitude largest?

(ii) Choose two functions $f(t)$ and $g(t)$ to show that $\mathcal{L}[f(t)g(t)] \neq \mathcal{L}[f(t)]\mathcal{L}[g(t)]$.

(iii) Find $\mathcal{L}^{-1}\left[\frac{1}{s^4}\right]$.

Problem 4. Match each differential equation on the left column with its corresponding general solution on the right. Be efficient, not much work it required.

(a) $y'' + y' + \frac{10}{4}y = \delta_0(t)$.

(b) $y'' + y = \sin(t)$.

(c) $y'' = 0$.

(d) $y'' + 4y' + y = 0$.

(i) $y(t) = k_1 e^{(-2-\sqrt{3})t} + k_2 e^{(-2+\sqrt{3})t}$.

(ii) $y(t) = k_1 t + k_2$.

(iii) $y(t) = k_1 e^{-t/2} \cos\left(\frac{3t}{2}\right) + k_2 e^{-t/2} \sin\left(\frac{3t}{2}\right) + u_0(t)\frac{2}{3}e^{-t/2} \sin\left(\frac{3t}{2}\right)$.

(iv) $y(t) = k_1 \sin(t) + k_2 \cos(t) - \frac{t}{2}\cos(t)$.

Problem 5. Consider the following non-linear system modelling the interaction between two species x and y:

$$\begin{cases} \frac{dx}{dt} = x(8 - 2x - y) = 8x - 2x^2 - xy \\ \frac{dy}{dt} = y(x^2 - y) = x^2y - y^2 \end{cases}$$

Since x and y are positive, you may consider the first quadrant only.

- Find the equations of the x- and y-nullclines and sketch them (use the graph paper provided).

- Find the equilibrium points that lie on the first quadrant and determine the type of equilibrium at each of them. Indicate them on the plot.

- Find the direction of the vector field in each region limited by the nullclines.

- If $x(0) > 0$ and $y(0) > 0$, what happens to both populations in the long term? Do species coexist nicely? Sketch a few solutions curves.

- Describe the evolution of species y in the absence of x, that is $y(0) > 0$ and $x(0) = 0$. Do the same for $x(0) > 0$ and $y(0) = 0$.

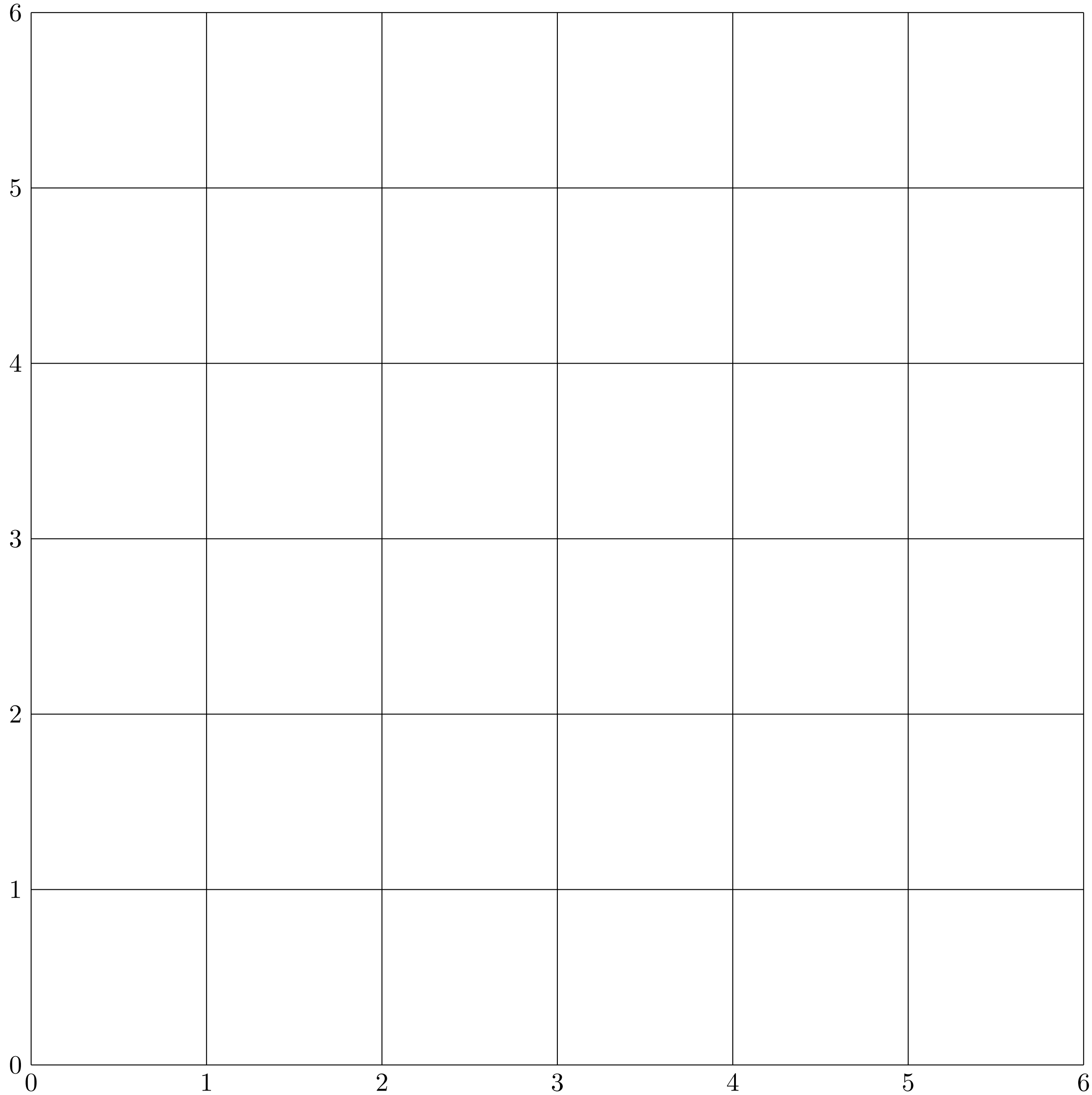

6
5
4
3
2
1
0
0 1 2 3 4 5 6

Table 6.1
Frequently Encountered Laplace Transforms.

$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$	$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a} \quad (s > a)$	$y(t) = t^n$	$Y(s) = \dfrac{n!}{s^{n+1}} \quad (s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \dfrac{\omega}{s^2 + \omega^2}$	$y(t) = \cos \omega t$	$Y(s) = \dfrac{s}{s^2 + \omega^2}$
$y(t) = e^{at} \sin \omega t$	$Y(s) = \dfrac{\omega}{(s-a)^2 + \omega^2}$	$y(t) = e^{at} \cos \omega t$	$Y(s) = \dfrac{s-a}{(s-a)^2 + \omega^2}$
$y(t) = t \sin \omega t$	$Y(s) = \dfrac{2\omega s}{(s^2 + \omega^2)^2}$	$y(t) = t \cos \omega t$	$Y(s) = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s} \quad (s > 0)$	$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$

Table 6.2
Rules for Laplace Transforms:
Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants α and a.

Rule for Laplace Transform	Rule for Inverse Laplace Transform
$\mathcal{L}\left[\dfrac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) = sY(s) - y(0)$	
$\mathcal{L}[y + w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$	$\mathcal{L}^{-1}[Y + W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$
$\mathcal{L}[\alpha y] = \alpha \mathcal{L}[y] = \alpha Y(s)$	$\mathcal{L}^{-1}[\alpha Y] = \alpha \mathcal{L}^{-1}[Y] = \alpha y(t)$
$\mathcal{L}[u_a(t) y(t-a)] = e^{-as} \mathcal{L}[y] = e^{-as} Y(s)$	$\mathcal{L}^{-1}[e^{-as} Y] = u_a(t) y(t-a)$
$\mathcal{L}[e^{at} y(t)] = Y(s-a)$	$\mathcal{L}^{-1}[Y(s-a)] = e^{at} \mathcal{L}^{-1}[Y] = e^{at} y(t)$