Below are the questions corresponding to the Final Exam. Please read the following instructions before starting:

- The exam must be submitted on Gradescope tomorrow, Thursday 7/2 by **11:59AM Boston Time**. You may upload a scan or a picture of you documents. Late submissions won't be accepted. Please indicate which pages correspond to which question when uploading.
- During the completion of this exam, you may **NOT** check the official textbook, notes, or any material posted on the course website.
- The use of any on-line/physical resources will be considered cheating.
- Each step on each answer must be justified. Indicate the methods you use to solve the equations (separable, guess and check, ...) to integrate function (u-substitution, by parts, partial fractions).
- By submitting the answers you agree to abide by the BU honor code. As instructors, we reserve our right, after the completion of the exam, to **ask privately** about your solutions to any of the problems.

Table 6.1
Frequently Encountered Laplace Transforms.

$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$	$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a} \quad (s > a)$	$y(t) = t^n$	$Y(s) = \dfrac{n!}{s^{n+1}} \quad (s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \dfrac{\omega}{s^2 + \omega^2}$	$y(t) = \cos \omega t$	$Y(s) = \dfrac{s}{s^2 + \omega^2}$
$y(t) = e^{at} \sin \omega t$	$Y(s) = \dfrac{\omega}{(s-a)^2 + \omega^2}$	$y(t) = e^{at} \cos \omega t$	$Y(s) = \dfrac{s-a}{(s-a)^2 + \omega^2}$
$y(t) = t \sin \omega t$	$Y(s) = \dfrac{2\omega s}{(s^2 + \omega^2)^2}$	$y(t) = t \cos \omega t$	$Y(s) = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s} \quad (s > 0)$	$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$

Table 6.2
Rules for Laplace Transforms:
Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants α and a.

Rule for Laplace Transform	Rule for Inverse Laplace Transform
$\mathcal{L}\left[\dfrac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) = sY(s) - y(0)$	
$\mathcal{L}[y + w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$	$\mathcal{L}^{-1}[Y + W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$
$\mathcal{L}[\alpha y] = \alpha \mathcal{L}[y] = \alpha Y(s)$	$\mathcal{L}^{-1}[\alpha Y] = \alpha \mathcal{L}^{-1}[Y] = \alpha y(t)$
$\mathcal{L}[u_a(t) y(t-a)] = e^{-as} \mathcal{L}[y] = e^{-as} Y(s)$	$\mathcal{L}^{-1}[e^{-as} Y] = u_a(t) y(t-a)$
$\mathcal{L}[e^{at} y(t)] = Y(s-a)$	$\mathcal{L}^{-1}[Y(s-a)] = e^{at} \mathcal{L}^{-1}[Y] = e^{at} y(t)$

1. (40 points) Consider the following non-linear system **restricted to the first quadrant** (i.e. $x \geq 0$ and $y \geq 0$).

$$\begin{aligned} \frac{dx}{dt} &= -5x^2 - xy + 114x \\ \frac{dy}{dt} &= -x^2y - y^3 + 900y \end{aligned}$$

   (a) (5 points) Find all of the equilibrium points in the first quadrant.

   (b) (5 points) Find the Jacobian matrix of the system.

   (c) (15 points) Find the linearized system near each equilibrium point and classify each equilibrium point.

   (d) (10 points) Sketch the nullclines and indicate the direction of the vector field along the nullclines.

   (e) (5 points) Finally, write a brief paragraph describing the possible behavior of a particular solution. You may choose the initial position, but it must be in the first quadrant, and it cannot be an equilibrium point.

2. (20 points) Answer the following questions about forced oscillators.

   (a) (10 points) Without solving explicitly, sketch an approximate graph of the solution with the given initial conditions.

      - $y'' + 4y = \delta_3(t)$, $y(0) = 0$, $y'(0) = 0$.
      - $y'' + y' + 4y = \delta_1(t)$, $y(0) = 3$, $y'(0) = 0$.
      - $y'' + 10y' + y = u_7(t)\sin(t)$, $y(0) = 3$, $y'(0) = 0$.

   (b) (10 points) In lecture we saw that, for a damped oscillator with sinusoidal forcing, the amplitude of the steady state was given by the following explicit formula:

$$A(p, q, \omega) = \frac{1}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}}.$$

      - Suppose $\omega = 2$ and $p = 1$ are fixed. For what value of q is the amplitude of the forced response largest?
      - For the above values, determine if slightly increasing ω has produces an increase or a decrease in $A(p, q, \omega)$. Justify your answer.

3. (20 points) In an RC circuit with certain parameters, the voltage across the capacitor $v_c(t)$ satisfies the differential equation

$$\frac{dv_c}{dt} + \frac{v_c}{2} = V(t),$$

   where $V(t)$ is the voltage across the source. The source comes with a switch. When off, the voltage is $V = 0$, and when on, $V = 1$.

(a) (10 points) Suppose that the system is switched off at $t = 0$. At $t = 1$, you decide to turn it on, and then at $t = 3$, you turn it off again. Write a closed form expression for $V(t)$. (HINT: you may write first a piece wise definition, then use one or more Heaviside functions $u_a(t)$).

(b) (10 points) Using the Laplace Transform method, find the solution with initial conditions $v_c(0) = 0$ and $v_c'(0) = 0$.

4. (30 points) True or False. Please justify your answers with sufficient details. A correct answer with no/not enough justification will receive no/partial credit(s).

   (a) (10 points) The phase plane of the system

$$\frac{dx}{dt} = 2x - x^2 - xy, \quad x \geq 0$$
$$\frac{dy}{dt} = y^2 - yx, \quad y \geq 0$$

   contains four separatrices.

   (b) (10 points) $\mathcal{L}[t^2]$ is undefined if $s > 0$.

   (c) (10 points) Consider the following system depending on the parameter a,

$$\frac{dx}{dt} = y - x^2,$$
$$\frac{dy}{dt} = y - ax.$$

   When $a > 5$, all the equilibrium points of the system are saddles.

5. (10 points) Evaluate the following expressions:

   (a) (5 points) $\mathcal{L}\left[e^{2t} t \sin(t)\right]$.

   (b) (5 points) $\mathcal{L}^{-1}\left[\frac{e^{3t}}{s^2-4}\right]$.