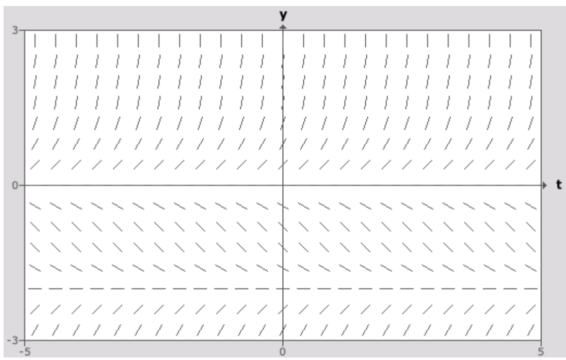
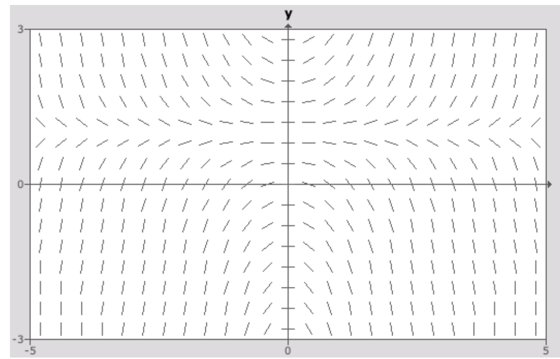
SOLUTIONS MIDTERM EXAM I

Problem 1. Indicate which slope field corresponds to each of the following differential equations:

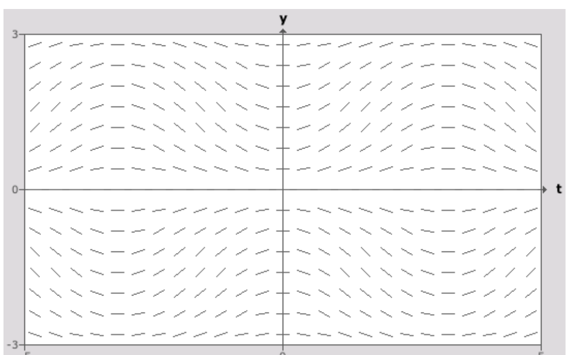
- $y' = y(y+2) \leftrightarrow (a)$
- $y' = ty - t \leftrightarrow (b)$
- $y' = e^t - 1 \leftrightarrow (d)$
- $y' = \sin(y)\sin(t) \leftrightarrow (c)$



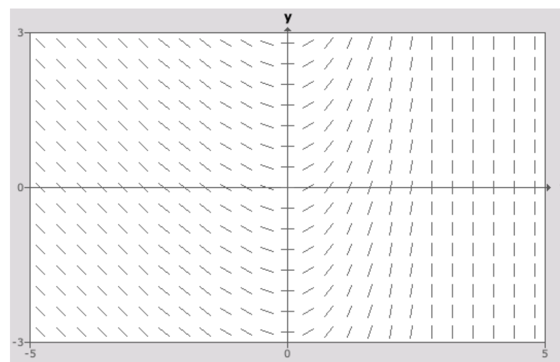
(a)



(b)



(c)



(d)

In each of the graphs above, sketch the solution with initial condition $y(0) = 1$.

Problem 2. Short questions:

- **T/F**: For the savings model $\frac{dM}{dt} = 0.1M - 300$, an initial amount of $M(0) = 7000$ leads to bankruptcy.

  FALSE: For $M > 300/0.1 = 3000$, the rate of change is always positive, so the amount of money $M(t)$ will actually increase to infinity.

- **T/F**: The differential equation $y' = y^3 + e^{\cos(t)}$ is linear.

  FALSE: It has a cubed term y^3.

- What is the correct guess for a particular solution $y_p(t)$ of $y' = 2y + 2\sin(-3t)$?

  The correct guess would be $y_p(t) = \alpha \sin(-3t) + \beta \cos(-3t)$.

- Find the bifurcation value in the one-parameter family of differential equations

  $$y' = (y + 1)(y - \mu).$$

  The equilibrium points are $y = -1$ and $y = \mu$. Therefore, it has 2 equilibria except in the case where $\mu = -1$. The bifurcation value is thus $\mu = -1$.

- For the equation $\frac{dy}{dt} = 2e^y - t$ and the initial value $y(2) = 0$. Estimate $y(2.1)$ using one step of Euler's Method.

  Since we use one step, the step size must be $\Delta t = 0.1$. Also, we compute $f(t_0, y_0) = f(2, 0) = 2e^0 - 2 = 0$. Therefore $y(2.1) \simeq y_0 + \Delta t f(t_0, y_0) = 0 + 0.1 \times 0 = 0$. The approximation is $y(2.1) = 0$.

Problem 3. Consider the autonomous differential equation

$$\frac{dy}{dt} = y^4 - 4y^2.$$

- Draw its phase line.

  First we find the equilibria: $y^4 - 4y^2 = 0$, so $y = 0, y = \pm 2$. Then, the sign of dy/dt is positive for $y > 2$ and $y < -2$ and negative for $2 > y > 0$ and $0 > y > -2$.

- Classify the equilibrium points as sinks, sources or nodes.

  There is a source at $y = 2$, a node at $y = 0$ and a sink at $y = -2$.

- Without explicitly solving the equation, what can you say about the long-term behaviour (forward and backward) of the solutions? Make sure you do so for enough initial conditions.

  It depends on the initial condition. In words:

  – For $y(0) = 2$, $y(0) = 0$ or $y(0) = -2$, $y(t)$ is a constant function.
  – For $y(0) > 2$: Forward: Either $\lim_{t\to\infty} y(t) = \infty$ or there is a τ such that $\lim_{t\to\tau^-} y(t) = \infty$. Backward: $\lim_{t\to-\infty} y(t) = 2$
  – For $2 > y(0) > 0$: Forward: $\lim_{t\to\infty} y(t) = 0$. Backward: $\lim_{t\to-\infty} y(t) = 2$
  – For $0 > y(0) > -2$: Forward: $\lim_{t\to\infty} y(t) = -2$. Backward: $\lim_{t\to-\infty} y(t) = 0$
  – For $-2 > y(0)$: Forward: $\lim_{t\to\infty} y(t) = -2$. Backward: Either $\lim_{t\to-\infty} y(t) = -\infty$ or there is a τ such that $\lim_{t\to\tau^+} y(t) = -\infty$

Problem 4. Find the solution of the initial value problem. Solve explicitly for $y(t)$.

$$\frac{dy}{dt} = t^2y^3 + 2ty^3, \qquad y(0) = 1.$$

Since it is separable, we use the method discussed in class:

$$\begin{aligned}
\frac{dy}{dt} &= t^2y^3 + 2ty^3, \\
\frac{1}{y^3}\frac{dy}{dt} &= t^2 + 2t, \\
\int \frac{1}{y^3}\frac{dy}{dt}dt &= \int \left(t^2 + 2t\right) dt, \\
\frac{-1}{2y^2} &= \frac{t^3}{3} + t^2 + C.
\end{aligned}$$

Plugging in the initial condition we find

$$\frac{-1}{2\times 1} = 0 + 0 + C \Rightarrow C = \frac{-1}{2}.$$

Finally, solving for y we have

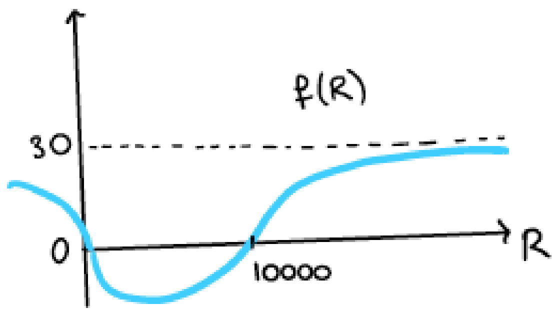
$$\begin{aligned}
2y^2 &= \frac{-1}{\frac{t^3}{3} + t^2 - \frac{1}{2}}, \\
y^2 &= \frac{-1}{\frac{2t^3}{3} + 2t^2 - 1}, \\
y &= \frac{1}{\sqrt{1 - \frac{2t^3}{3} - 2t^2}}.
\end{aligned}$$

Note that we choose the positive square root because $y(0) = 1 > 0$.

Problem 5. The city of Bollston is infested with rats. Its population $R(t)$ can be modelled by an autonomous differential equation

$$\frac{dR}{dt} = f(R),$$

where the graph of $f(R)$ is given. The values on the graph are **approximate**, no calculations required.



- Draw the phase line associated to this equation.

  From the graph of $f(R)$ we see that there are two equilibrium points at $R = 0$ and $R = 10000$. Moreover, dR/dt is positive for $R < 0$ and $R > 10000$, and negative for $0 < R < 10000$.

- If the initial population of rats is $R(0) = 5000$, what happens in the long term? And what if $R(0) = 15000$? After what value of $R(t)$ the population grows without control?

  From the phase line, we can deduce that is $R(0) = 5000 < 10000$, the rat population will decrease asymptotically to 0. Similarly, if $R(0) = 15000 > 10000$, the rat population will increase to infinity at a rate of approximately 30. The threshold is at the equilibrium point $R = 10000$.

- To save the city, the department of homeland security decides to invest money and resources. These actions introduce a parameter μ in the equation:

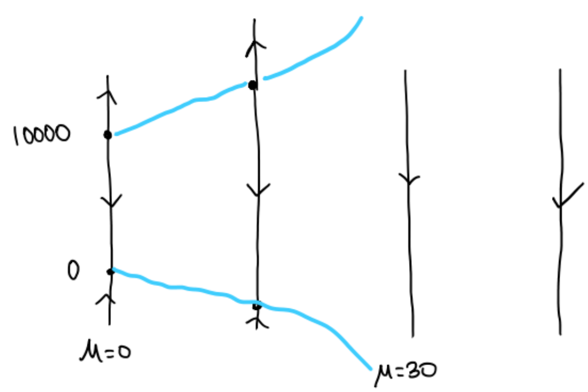
$$\frac{dR}{dt} = f(R) - \mu.$$

What is the minimum value of μ, so that no matter what the initial population of rats is, they will always become extinct?

From the graph of $f(R)$, we can see that for $\mu = 30$, $f(R) - \mu$ is always negative. For $\mu < 30$, there is one positive equilibrium point, thus the minimal value is $\mu = 30$.

- Draw the bifurcation diagram.

For $\mu \geq 30$, it shouldn't show any equilibrium point > 0. Since it isn't clear from the graph what happens $R < 0$, any answer regarding that range was considered correct.



Problem 6. Solve the following initial value problem:

$$y' + \frac{y}{t} = e^t - \frac{3}{t}, \qquad y(1) = 0.$$

It is a first order linear ODE with non-constant coefficient and not homogeneous. We can find a solution by using an integrating factor $\mu(t)$. Be definition

$$\mu(t) = e^{\int \frac{1}{t}} = e^{\ln(t)} = t.$$

Multiplying by $\mu = t$ and rewriting as product rule:

$$\begin{aligned} ty' + y &= te^t - 3, \\ \frac{d}{dt}(yt) &= te^t - 3, \\ yt &= \int (te^t - 3), \\ yt &= te^t - e^t - 3t + C, \\ y &= e^t - \frac{e^t}{t} - 3 + \frac{C}{t}. \end{aligned}$$

Plugging in the initial condition we can determine C:

$$\begin{aligned} 0 &= e^1 - \frac{e^1}{1} - 3 + \frac{C}{1}, \\ 0 &= -3 + C, \qquad C = 3. \end{aligned}$$

The solution is

$$y(t) = (e^t - 3)\left(1 - \frac{1}{t}\right).$$

BONUS: Suppose that the total American population in the year t is given by the function $P(t)$. Moreover, let $R(t)$ be the number of American citizens that identify as Republican in the year t, and $D(t)$ the Democrats. People's opinion can change, but ALL citizens identify as either, meaning

$$R(t) + D(t) = P(t).$$

- If the population $P(t) = P$ is constant, how are dD/dt and dR/dt related?

  All the relations follow from the above equation and its differentiation:

  $$\frac{dR}{dt} + \frac{dD}{dt} = P'(t),$$

  and since P is constant,

  $$\frac{dR}{dt} + \frac{dD}{dt} = 0.$$

- Suppose that $D(t)$ satisfies the equation $dD/dt = f(t)$, and that $P(t)$ and $f(t)$ are unknown functions. Write a differential equation for $R(t)$.

  $$\frac{dR}{dt} = P'(t) - \frac{dD}{dt} = P'(t) - f(t),$$

  $$\frac{dR}{dt} = P'(t) - f(t).$$

- Suppose now that $R(t)$ satisfies $dR/dt = g(t, R)$. Write a differential equation for $D(t)$ not containing the variable R.

  $$\frac{dD}{dt} = P'(t) - \frac{dR}{dt} = P'(t) - g(t, R) = P'(t) - g(t, P - D),$$

  $$\frac{dD}{dt} = P'(t) - g(t, P - D).$$