1. (a) (10 points) Say all you can about the following differential equation **without** solving it.

$$\frac{dy}{dt} = y^2 - 5y + 6.$$

This is a first-order, autonomous ODE. It has two equilibrium points $y = 3$ and $y = 2$, being a source and a sink respectively. From the picture of the phase line, we can deduce that:

- As $t \to \infty$, if $y(0) < 3$ $y \to 2$, and if $y(0) > 3$ then $y \to \infty$.
- As $t \to -\infty$, if $y(0) < 2$ $y \to -\infty$, and if $y(0) > 2$ then $y \to 3$.

(b) (10 points) Use Euler's Method to approximate the solution whose initial value is $y(0) = \frac{3}{2}$ with a step size of $\Delta t = 1$ for 2 steps, and a step size of $\Delta t = \frac{1}{2}$ for 4 steps. Sketch your two approximations. Which makes more sense and why?

2. (20 points) A 100-gallon tank initially contains 25 gallons of salt water containing 5 pounds of salt. Suppose salt water containing 1 pound of salt per gallon is pumped into the top of the tank at the rate of 3 gallons per minute, while a well-mixed solution leaves the bottom of the tank at a rate of 2 gallon per minute. Write an initial value problem that models the amount of salt in the tank. How much salt is in the tank when the tank is full? Indicate the units you are working with.

3. (20 points) Consider the Ermentrout-Kopell model for the spiking of a neuron

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta)I$$

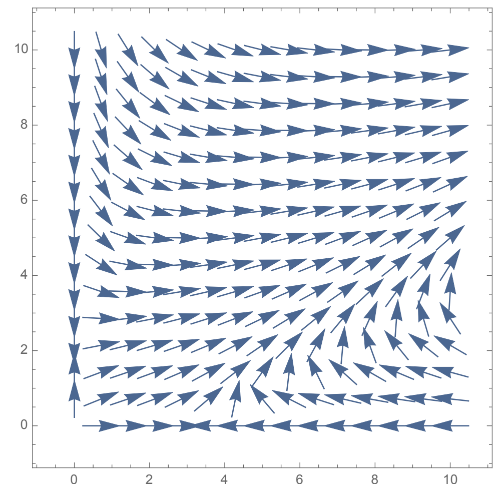
where we suppose that the input I is a constant. Describe the bifurcation(s) that occur as the parameter I varies. Draw the phase lines for values of the parameter slightly smaller than, slightly larger than, and at the bifurcation value(s).

4. (20 points) The following model represents the interaction between two species x and y.

   (a) (10 points) Describe the main features of the model, and come up with a story/situation, describing what species x and y may represent, and what type of interaction they have. Indicate the equilibrium points. (Hint: you may want to rewrite the equations in a more standard way).

$$\begin{cases} \frac{dx}{dt} &= 2x\left(1 - \frac{x-2y}{3}\right) \\ \frac{dy}{dt} &= y\left(1 - \frac{y-x}{2}\right) \end{cases}$$

   (b) (5 points) Sketch the solution curve with initial condition $(x_0, y_0) = (f, l)$, where f and l are the first and last digits of your BU ID. You are provided with the following direction field.



   (c) (5 points) Sketch the corresponding $x(t)$ and $y(t)$ graphs.

5. (20 points) Solve the following initial value problems. Please indicate all the steps and methods as you use them.

   (a) (10 points) $\frac{dy}{dt} = 2y + e^{-t} + 3\cos(2t), \qquad y(0) = 3.$

   (b) (10 points) $\frac{dy}{dt} = \frac{t+y^3 t}{y^2}, \qquad y(0) = 2.$

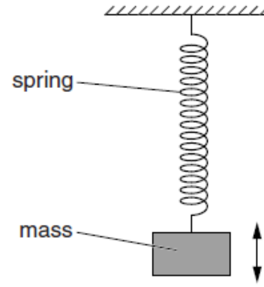
6. (20 points) True or False. Short answers. A correct answer with no justification will be considered incorrect.

   (a) (5 points) We can assure that there exists a unique solution to $y' = \sqrt{y}$ through $y(0) = 2$.

   (b) (5 points) Your bank account contains \$100000 and generates 5% interest annually compounded continuously. Can you live off your savings forever if you spend approximately \$4500 a year?

   (c) (5 points) A nonhomogeneous linear equation cannot be separable.

   (d) (5 points) If $y_1(t)$ solves $\frac{dy}{dt} = a(t)y + b_1(t)$ and $y_2(t)$ solves $\frac{dy}{dt} = a(t)y + b_2(t)$, then $y_1(t) + y_2(t)$ solves $\frac{dy}{dt} = a(t)y + b_1(t) + b_2(t)$.



7. (10 points) Consider a vertical mass-spring system shown in the figure above. Given that we only take into account the forces due to gravity and the spring as the forces acting on the mass, construct a model describing the position of the mass with respect to time and find the length of the spring, expressed in terms of unknown coefficients, when the mass is in its rest position. Please specify all the variables and coefficients you use in your model. (Some coefficients to be considered: the mass, gravity, the spring constant, the rest length of the spring.)