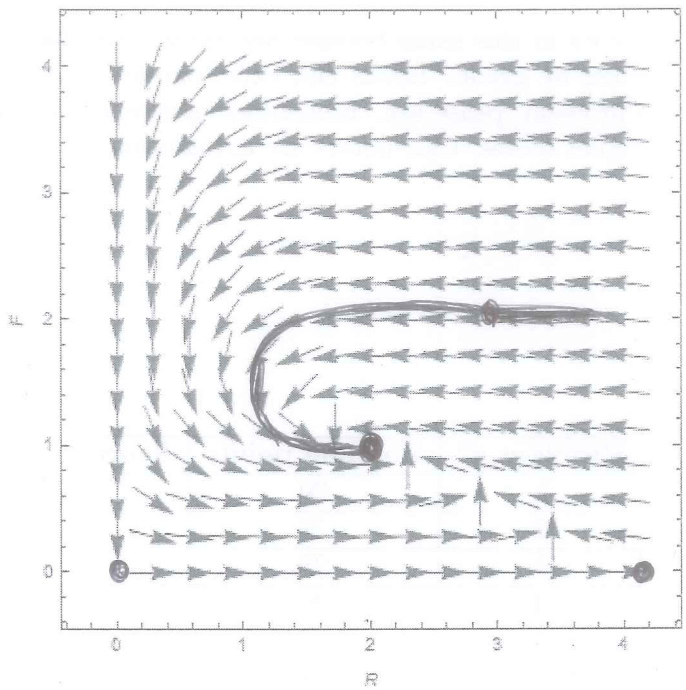
Problem 1. The following system of differential equations models the interaction of foxes (F) and rabbits (R) in the forests of Massachusetts.

$$\begin{cases} \frac{dR}{dt} = 4R\left(1-\frac{R}{4}\right) - 2RF \\ \frac{dF}{dt} = -F + \frac{RF}{2} \end{cases}$$

You are also given a plot of the associated slope field



- Compute the equilibrium points AND plot them on the graph above.

(i) $4R\left(1-\frac{R}{4}\right) - 2RF = 0,$

(ii) $-F + \frac{RF}{2} = 0, \Rightarrow F\left(\frac{R}{2}-1\right) = 0$, $\nearrow F=0$, $\searrow R=2$

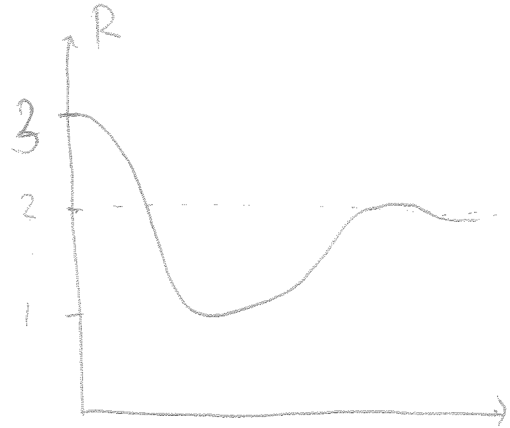
If $F=0$, then (i) $4R\left(1-\frac{R}{4}\right)=0$ so $R=0$, $R=4$

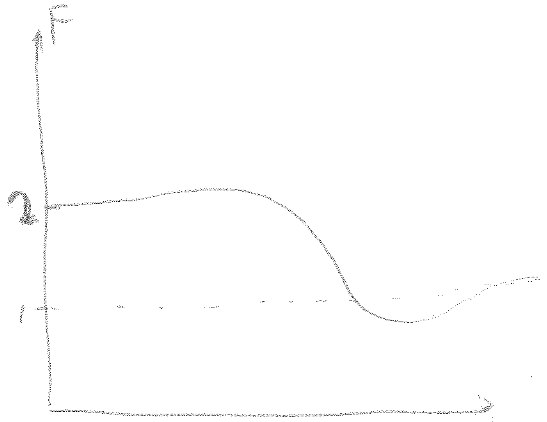
If $R=2$, then (i) $4-4F=0$, so $F=1$

Points: $(0,0)$, [illegible], $(2,1)$ $(4,0)$

- Sketch the solution curve with initial conditions $R(0) = 3$ and $F(0) = 2$.

- For this solution, sketch the functions $R(t)$ and $F(t)$ on two separate graphs.





- Describe what happens with both populations in the long term (forward time).

The population stabilize and approach the eq. point at (2,1).

Problem 2. Consider the linear system of differential equations

$$\frac{dY}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \cdot Y.$$

- Find the corresponding eigenvalue and eigenvector.

· Char. poly: $\lambda^2 - 6\lambda + 9 = (\lambda-3)^2 = 0$, $\lambda = 3$ repeated.

· Eigenvector

$$\begin{pmatrix} 2-3 & 1 \\ -1 & 4-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad , -x+y=0 \, , \quad V_1 = (1,1)$$

- Find the general solution.

$Y_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is one straight line sol.

for the other, we find $\overset{(x_0,y_0)}{V_0}$ such that

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \text{ so } \quad \begin{matrix} -x_0 + y_0 = 1 \\ V_0 = (0,1) \end{matrix}$$

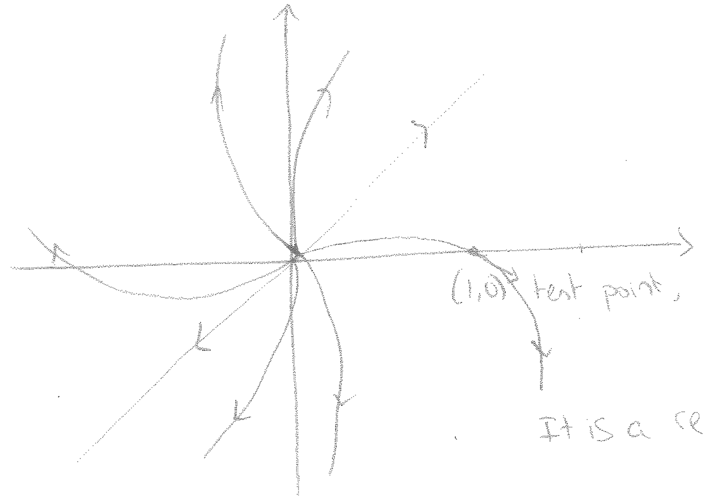
now we write

$$Y_2(t) = te^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the gueal sol'n is thus:

$$Y(t) = k_1 Y_1(t) + k_2 Y_2(t) = k_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2\left(te^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

- Sketch the phase portrait of this system and indicate the type of equilibria.



(1,0) test point, $\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

It is a repeated eigenvalue.

- Find the solution with initial condition $Y(0) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

using the general solution from before, we have

$$Y(0) = k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\begin{cases} k_1 = 5 \\ k_1 + k_2 = 2 \end{cases}, \text{ so } k_2 = -3$$

$$Y(t) = 5e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3\left(te^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= -3te^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Problem 3. TRUE or FALSE. (No justification needed)

- Every linear system always has at least one equilibrium point.

TRUE: at (0,0).

- The equation $3y'' - 5y' + 7y = 0$ models a harmonic oscillator.

FALSE: $b=-5$, can't be negative.

- In a linear system with two real eigenvalues $\lambda_2 > \lambda_1 > 0$, the solution curves near $(0,0)$ are tangent to the line with eigenvalue λ_2.

FALSE; $\underline{k_2 e^{\lambda_2 t} v_2} + k_1 e^{\lambda_1 t} v_1$

decreases faster as $t \to -\infty$.

- In the SIR model of an epidemic. The values $(S, 0)$ are equilibrium points for any S. Recall that S is the ratio of susceptible people, I the ratio of infected and R, recovered.

TRUE: No infected people to spread the disease.

- Differential equations are really useful.

TRUE.

Problem 4. Consider the system of differential equations

$$\frac{dY}{dt} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \cdot Y.$$

- Find the corresponding eigenvalues.

Char poly: $\lambda^2 - 4\lambda + 13$, $\lambda = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$

Eigenvector (only need one)

$$\begin{pmatrix} 2-(2+3i) & -3 \\ 3 & 2-(2+3i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad -3ix - 3y = 0$$

$V = (i, 1)$

- Find the general solution of this system.

Complex solution: $Y(t) = e^{(2+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}$

write it as real $+ i$ imaginary.

$$Y(t) = e^{2t}(\cos 3t + i \sin 3t) \begin{pmatrix} i \\ 1 \end{pmatrix}$$

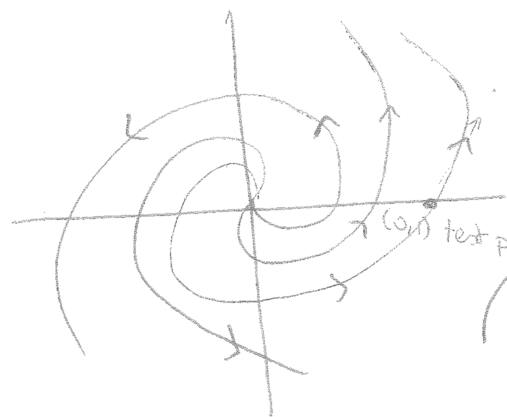
$$= e^{2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + i e^{2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$$

so

$$\underline{Y}(t) = k_1 e^{2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + k_2 e^{2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$$

- Sketch the phase portrait of this system and indicate the type of equilibria.

since $\lambda = 2 \pm 3i$, and $2 > 0$, it is a spiral source.



counterclockwise

(0,1) test point for orientation

$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

- Indicate the period of the oscillations, if any.

the angular frequency is $\omega = 3$, so

the period is $\frac{2\pi}{\omega} = \frac{2\pi}{3}$ //

Problem 5. The brakes of a brand new car can be modelled as a damped harmonic oscillator with mass $m = 1$, spring constant $k = 3$ and variable damping b.

- Write the second order equation for this oscillator and the matrix of the corresponding system in terms of b.

  - Second order: $y'' + by' + 3y = 0$
  - System: $\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -3y - bv \end{cases}$, so $\begin{pmatrix} 0 & 1 \\ -3 & -b \end{pmatrix}$

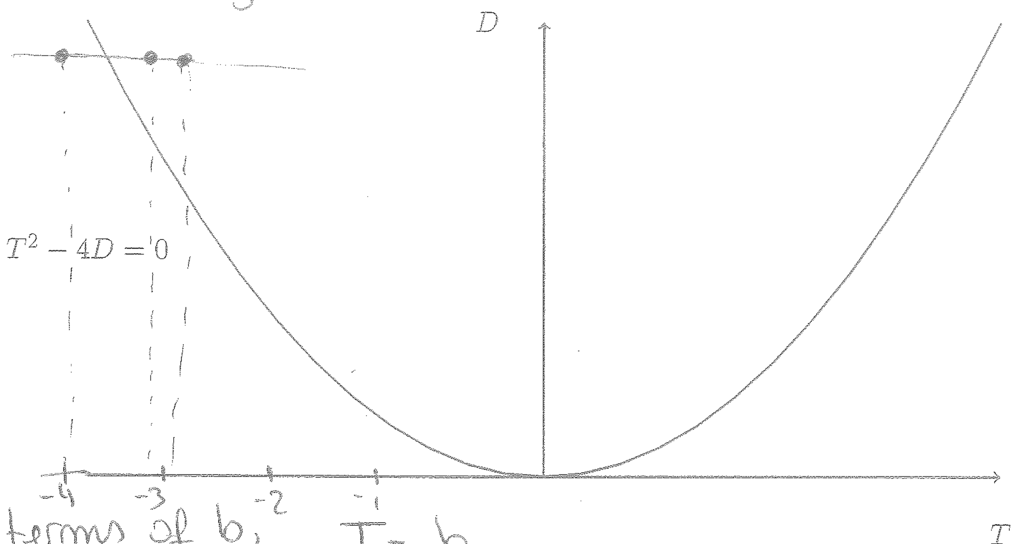
- At the time the car is bought, the damping is $b = 4$. Classify the type of oscillator, and justify why this makes sense from a real life perspective.

If $b=4$, then $\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$ has $T = -4$, $D = 3$

and $T^2 - 4D = 16 - 12 = 4 > 0$, so it is an overdamped oscillator.

- Over time, due to usage, the brake system wears down. This can be modelled by a decreasing damping coefficient $b(t) = 4 - t/4$, where t is the time in years. Plot in the Trace-Determinant plane the points corresponding to the system at the times $t = 0$, $t = 3$ and $t = 4$.

they lie in the line $D=3$.



In terms of b, $T=-b$, $D=3$ so we have:

$\underline{t=0}$, $b(0)=4$, so $T=-4$, $D=3$ $\quad T^2-4D = 4 > 0$

$\underline{t=3}$, $b(3)=4-3/4$, so $T=-13/4$, $D=3$ $\quad T^2-4D = \frac{169}{16}-12 < 0$

$\underline{t=4}$, $b(4)=4-1$, so $T=-3$, $D=3$ $\quad T^2-4D = 9-12 < 0$

- Find the values of t and b when the bifurcation occurs. In other words, after how many years should the brakes be changed?

the bifurcation occurs when $T^2=4D$, that is

$$(-b)^2 = 4 \cdot 3 \Rightarrow b = 2\sqrt{3}$$

solving for t,

$b(t) = 4 - t/4 = 2\sqrt{3}$, $t = 16 - 8\sqrt{3} = \underline{8(2-\sqrt{3})}$ $\underline{\text{years}}$. ≈ 2.14

after this point the oscillator will be <u>underdamped</u>

BONUS

Consider the following system of ODEs with three dependent variables x, y and z.

$$\begin{cases} \frac{dx}{dt} = 2x + z \\ \frac{dy}{dt} = 2y + 3z \\ \frac{dz}{dt} = x + y + z \end{cases}$$

- Write the associated matrix.
- Check that the vector $v = (-1, 1, 0)$ is an eigenvector.
- Find its eigenvalue.
- From your previous calculations, write a straight line solution of the system.

- $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ it is a 3x3 matrix

- check that $A \cdot v = \lambda v$:

eigenvalue

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -1+1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

- $\lambda = 2$

- $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{2t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, straight line solution.