1.5 Existence and Uniqueness

In this section we will answer two questions:

- When do solutions to an initial value problem exist?
- When are solutions to an initial value problem unique?

In the context of quadratic equations you already know the answer. The equation

$$X^2 + bX + c = 0$$

has **two** distinct (real $\mathbb{R}$) solutions if the discriminant $b^2 - 4c > 0$, there is **one** unique solution when $b^2 - 4c = 0$ and there are **zero** (real) solutions if $b^2 - 4c < 0$. For differential equations, it is a little bit more subtle.

Existence

Consider the IVP

$$\frac{dy}{dt} = \frac{y}{t^2}, \qquad y(0) = 2.$$

- Try solving it, what goes wrong?

- Read the following theorem and explain how it applies to this problem.

Theorem (Existence)**.** *Suppose $f(t, y)$ is a continuous function in a rectangle of the form $\{(t, y) \mid a < t < b, c < y < d\}$ in the ty-plane. If (t_0, y_0) is a point in this rectangle, then there* ***exits*** *an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ that solves the initial-value problem*

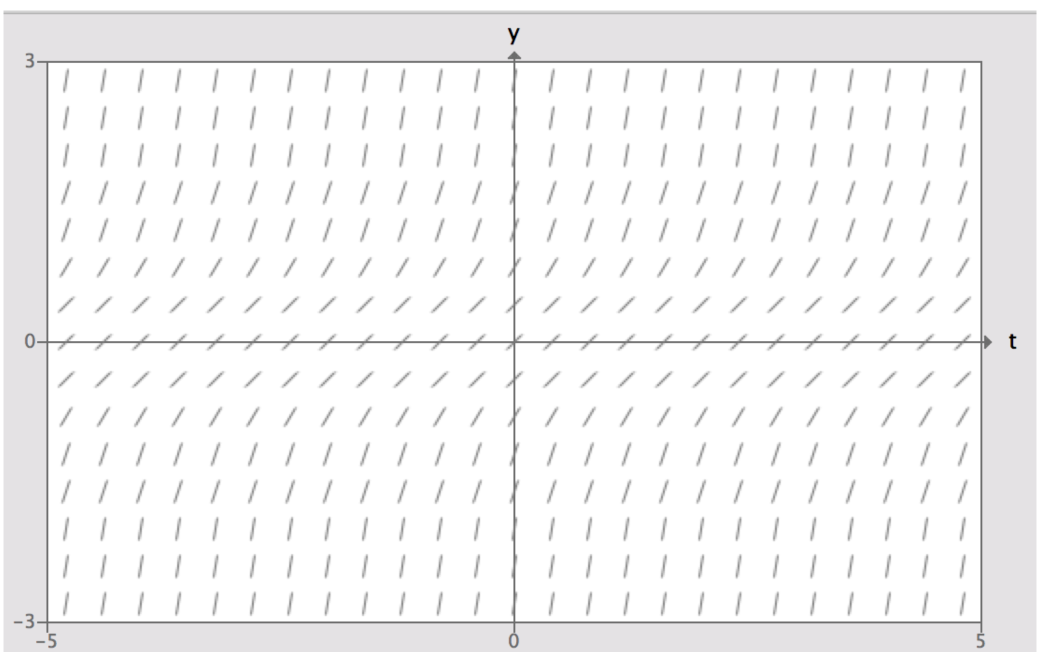
$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

Consider now the following IVP

$$\frac{dy}{dt} = 1 + y^2, \qquad y(0) = 0.$$

- Justify why a solution $y(t)$ exists. Find $y(t)$. (Hint: $\int 1/(1+y^2) = \arctan(y) + C$)

- Plot the solution and check that it matches the slope field below



- What is the domain of this function? What is the value of ϵ in this example? In other words, how long can the solution be extended?

Uniqueness

Consider the IVP

$$\frac{dy}{dt} = 3y^{2/3}, \qquad y(0) = 0.$$

- Check that the functions $y_1(t) = 0$ and $y_2(t) = t^3$ both are solutions to the IVP.

- Plot both solutions.

Theorem (Uniqueness)**.** *Suppose $f(t, y)$ and $\partial f/\partial y$ are continuous functions in a rectangle of the form $\{(t, y) \mid a < t < b, c < y < d\}$ in the ty-plane. If (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial value problem,*

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

for all t in the interval $t_0 - \epsilon < t < t_0 + \epsilon$ where ϵ is a positive number, then

$$y_1(t) = y_2(t)$$

*for $t_0 - \epsilon < t < t_0 + \epsilon$. That is, the solution to the I.V.P. is **unique**.*

- Explain how the theorem can be applied to the previous problem.

Some remarks

Sometimes, it is tricky to fully understand the scope of a theorem. The two theorems in this section give **sufficient** conditions for an IVP to have unique solutions. However, this conditions are not **necessary**.

Given a differential equation, you **can** tell if it has a solution for a certain initial condition, and if this solution is unique. However, you **can't** tell what the domain of this solution will be, or if it will blow up to infinity in finite time.

For more information, you can read section 1.5 in the textbook. The problems 1,4,6,7,8,9,10,11 and 14 of this section are specially interesting.