Counting Geometric Objects

An introduction to Enumerative Geometry

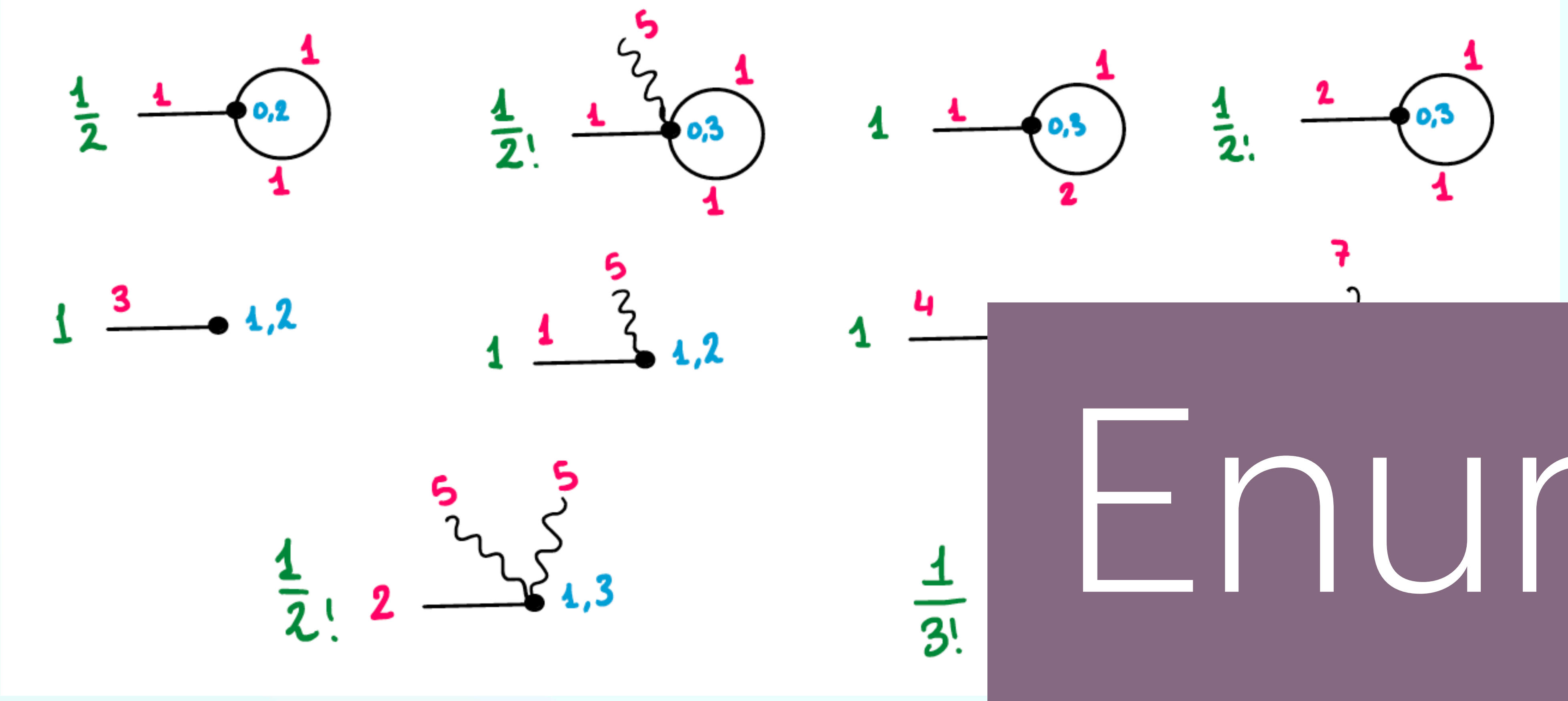
Roderic Guigo Corominas, Summer 2024

About myself

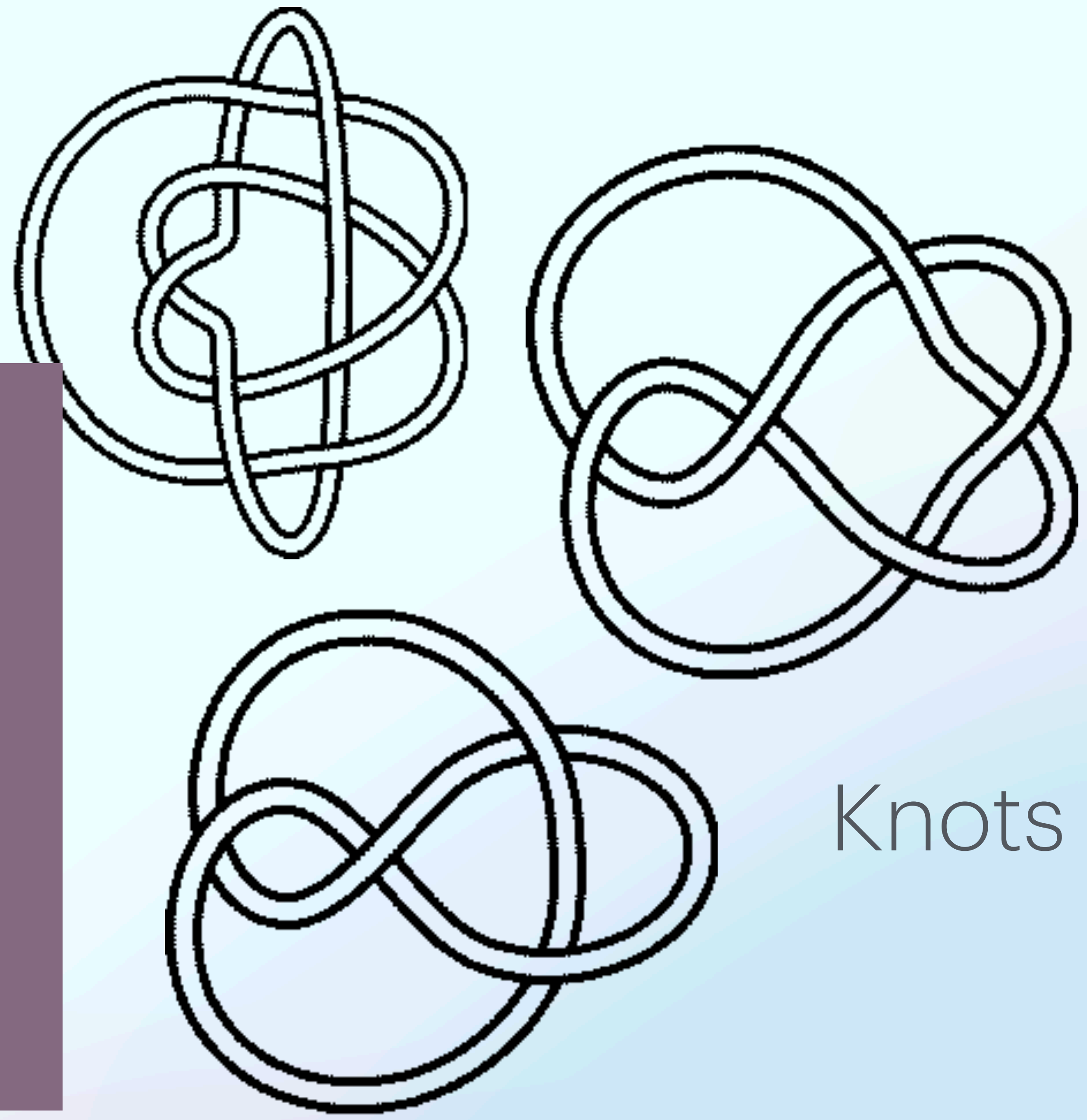
- Studied mathematics & physics at the University of Barcelona (UB)
- Got a PhD in mathematics from Boston University (BU)
- Currently teach mathematics at Harvard University. College courses for undergraduates and in the Pre-College program

My Thesis

Non-Perturbative Topological Recursion and Knot Invariants

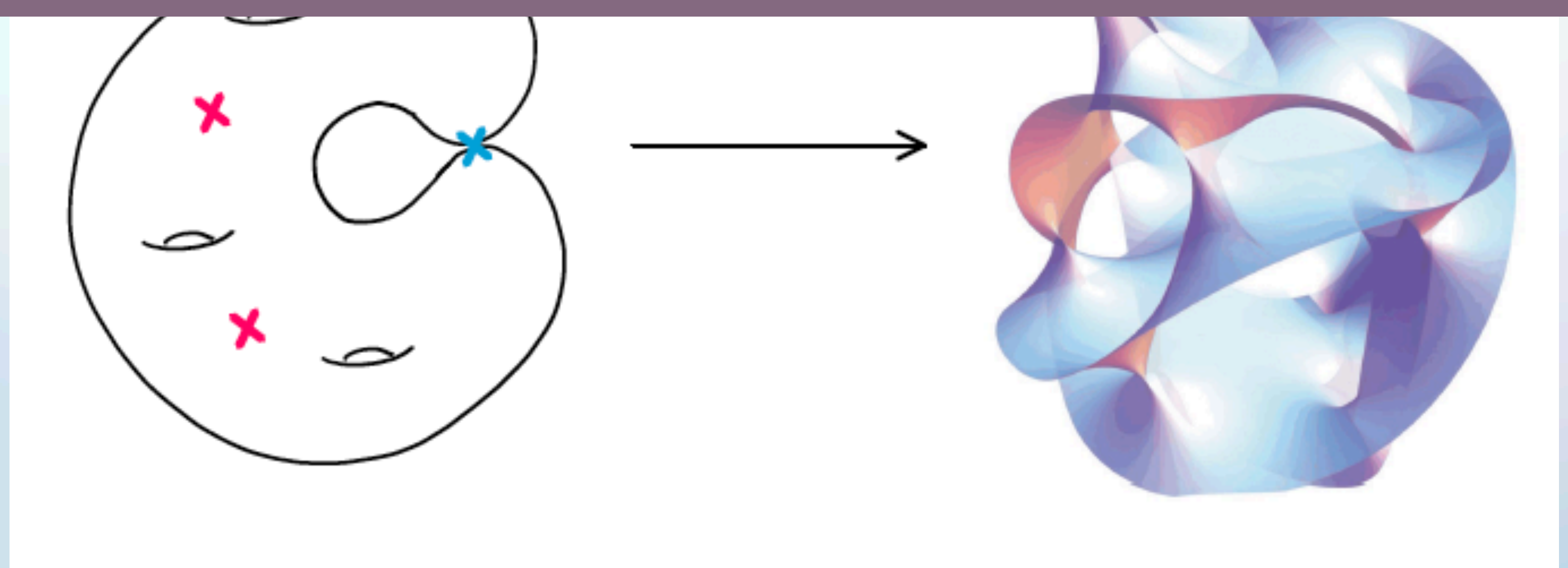


Graphs



Knots

Enumerative Geometry



Curves

Enumerative Geometry

The history of its problems...

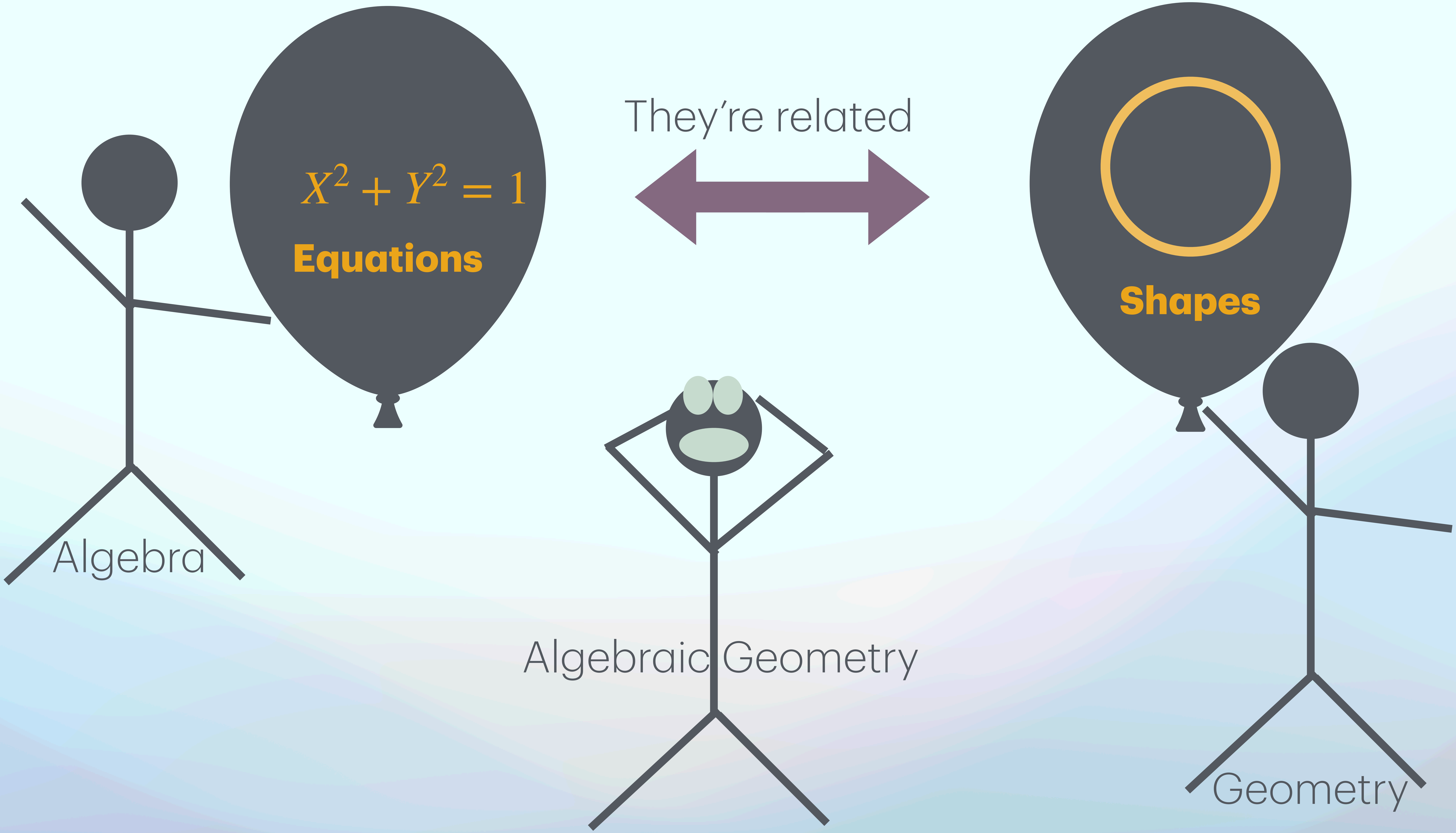
- Ancient Greece: Apollonius of Perga (problem of Apollonius)
- 18th century: Étienne Bézout (intersections of polynomials)
- 19th century: term "Enumerative Geometry" is coined (first as Géométrie Énumérative). Michel Chasles (Chasles Theorem), Hermann Schubert (Schubert Calculus)
- 20th century: David Hilbert (15th of his famous 23 problems)
- Late 20th century: Quantum Cohomology, etc.

Enumerative Geometry

…and discoveries that helped it develop

- Ancient Greece: Euclid (foundational work in *The Elements*), Archimedes
- 17th and 18th centuries: Pierre de Fermat (method of *Adequality*), Rene Descartes (introduced coordinate geometry), Isaac Newton, Gottfried Wilhelm Leibniz (calculus)
- 19th century: Bernhard Riemann (Riemann surfaces)
- 20th Century: Andre Weil, Alexander Grothendieck (algebraic geometry), William Fulton (intersection theory)
- Late 20th Century: Edward Witten, Maxim Kontsevich

a branch of Algebraic Geometry



and hence a branch of Geometry

… from ChatGPT

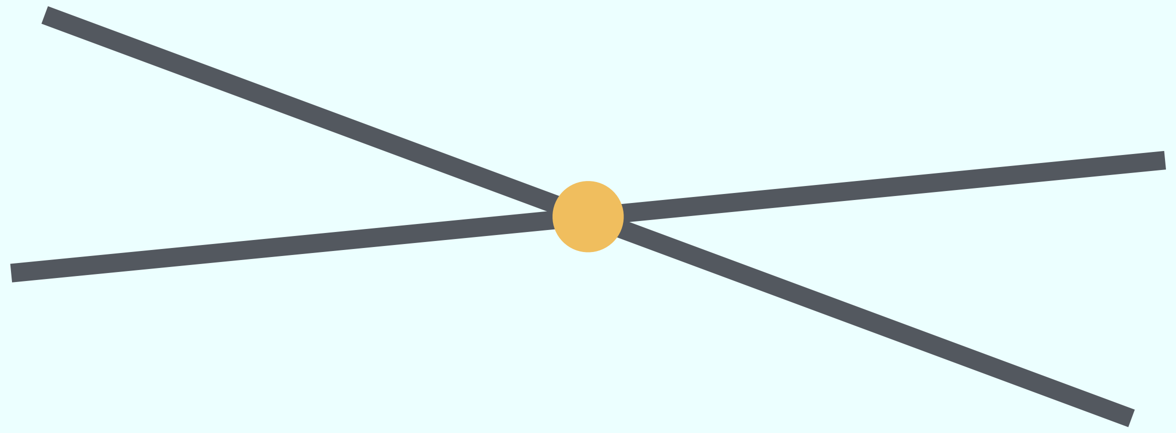
Geometry is a branch of mathematics that studies the **properties**, relationships, and measurements of points, lines, shapes, surfaces, and solids. It deals with the spatial properties of **objects** and their arrangements, focusing on their sizes, shapes, relative positions, and the properties of **space**.

"The Study of Properties of Objects in Space"

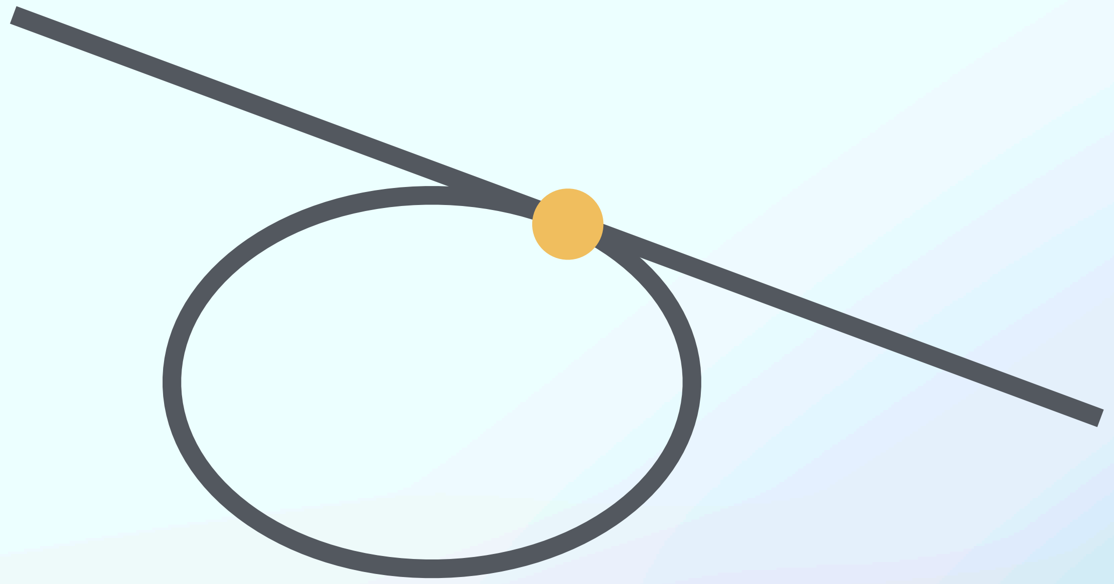
Points, Lines and Circles

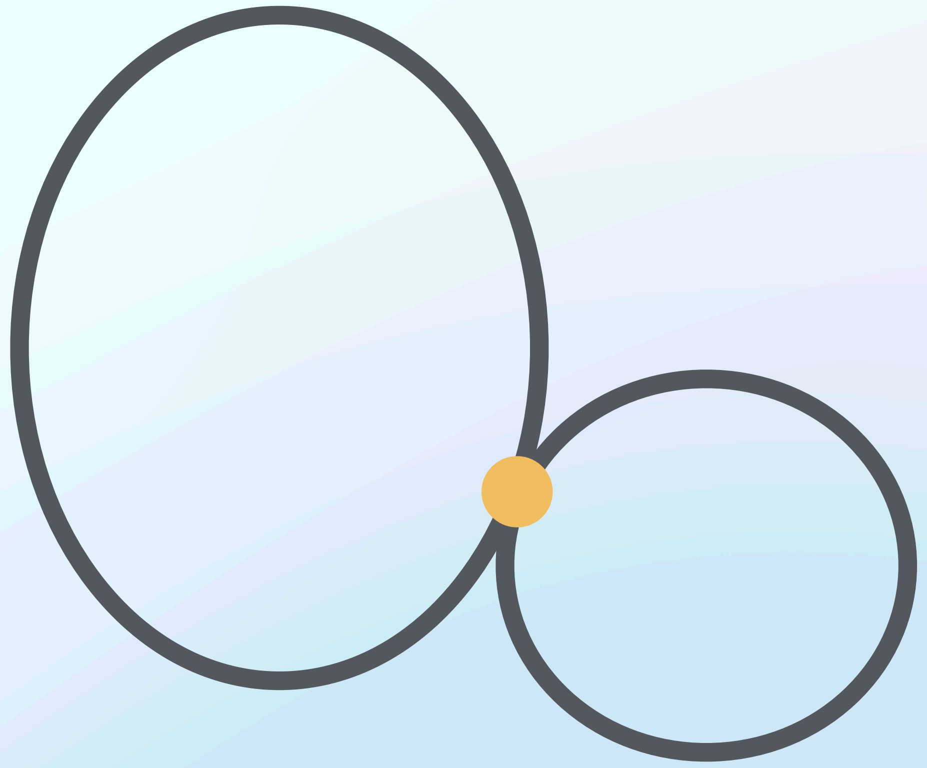
Some Preliminary Concepts

- **Intersecting** shapes: meet at a point

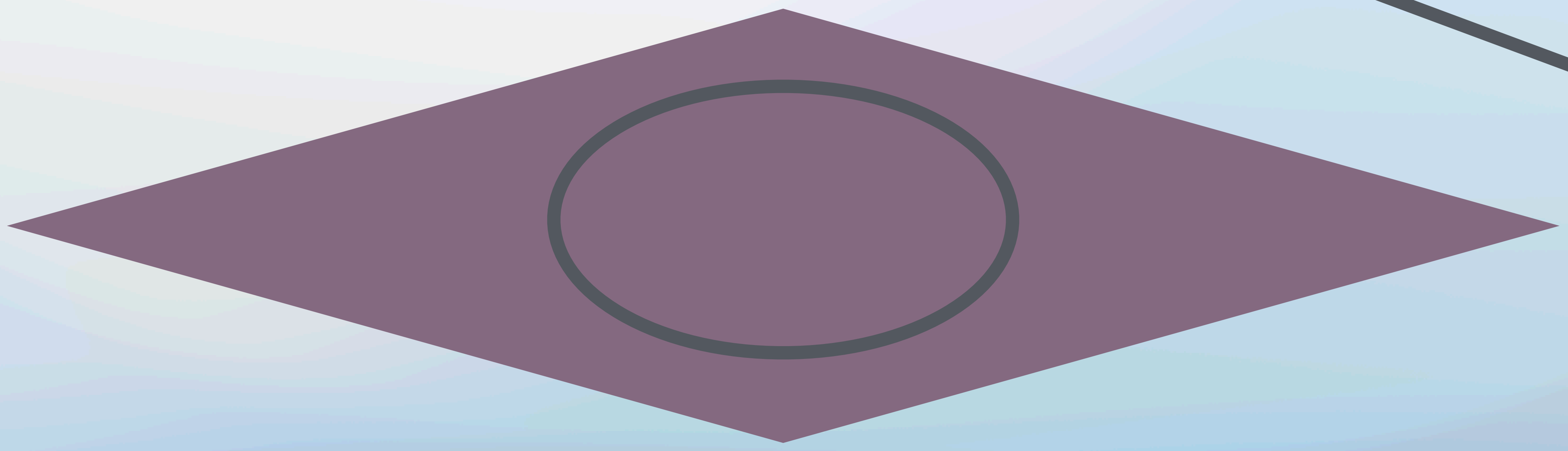


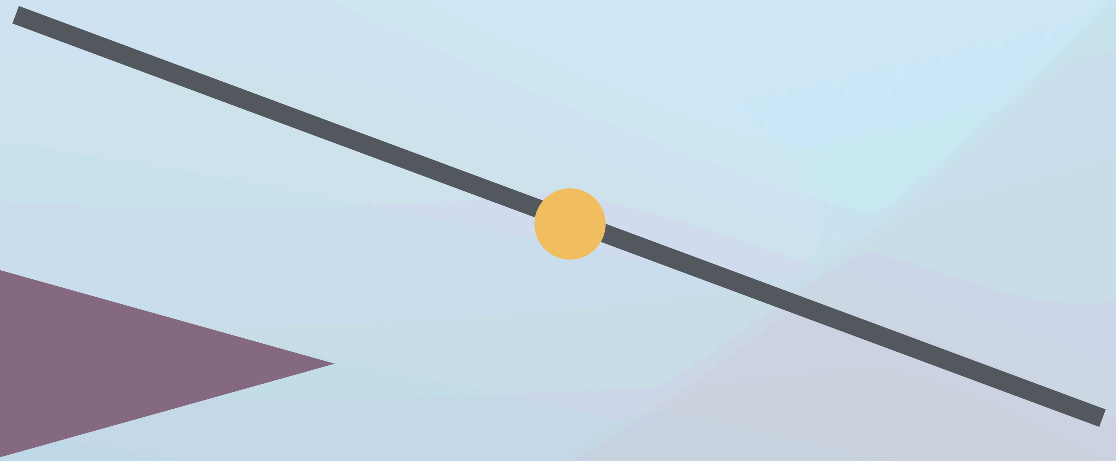
- **Tangent** shapes: intersect and have the same slope





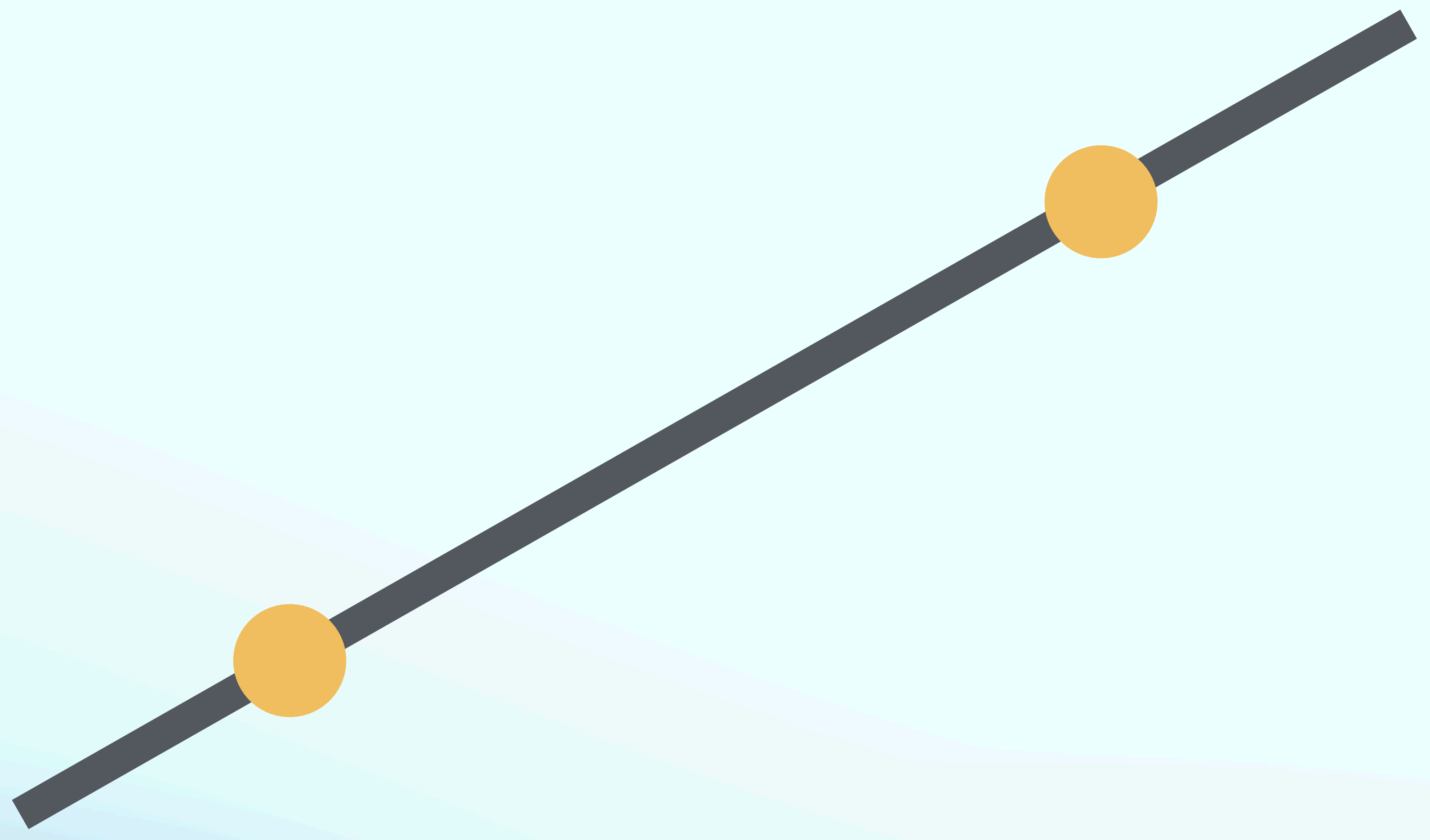
- **Contained** shapes: one is inside the other



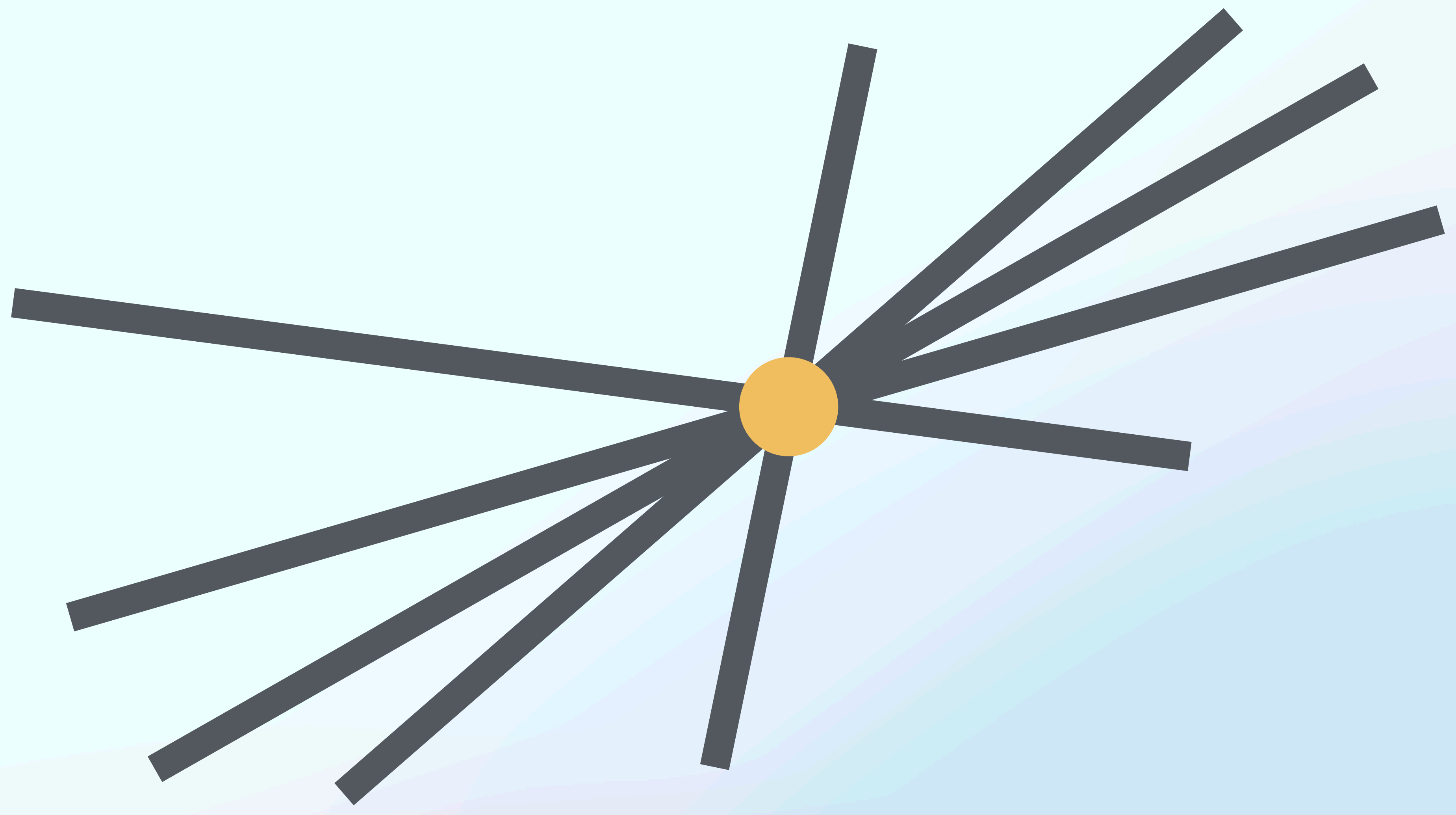


Lines Through Two Points

How many lines can be drawn through two points?



One, if the points are different

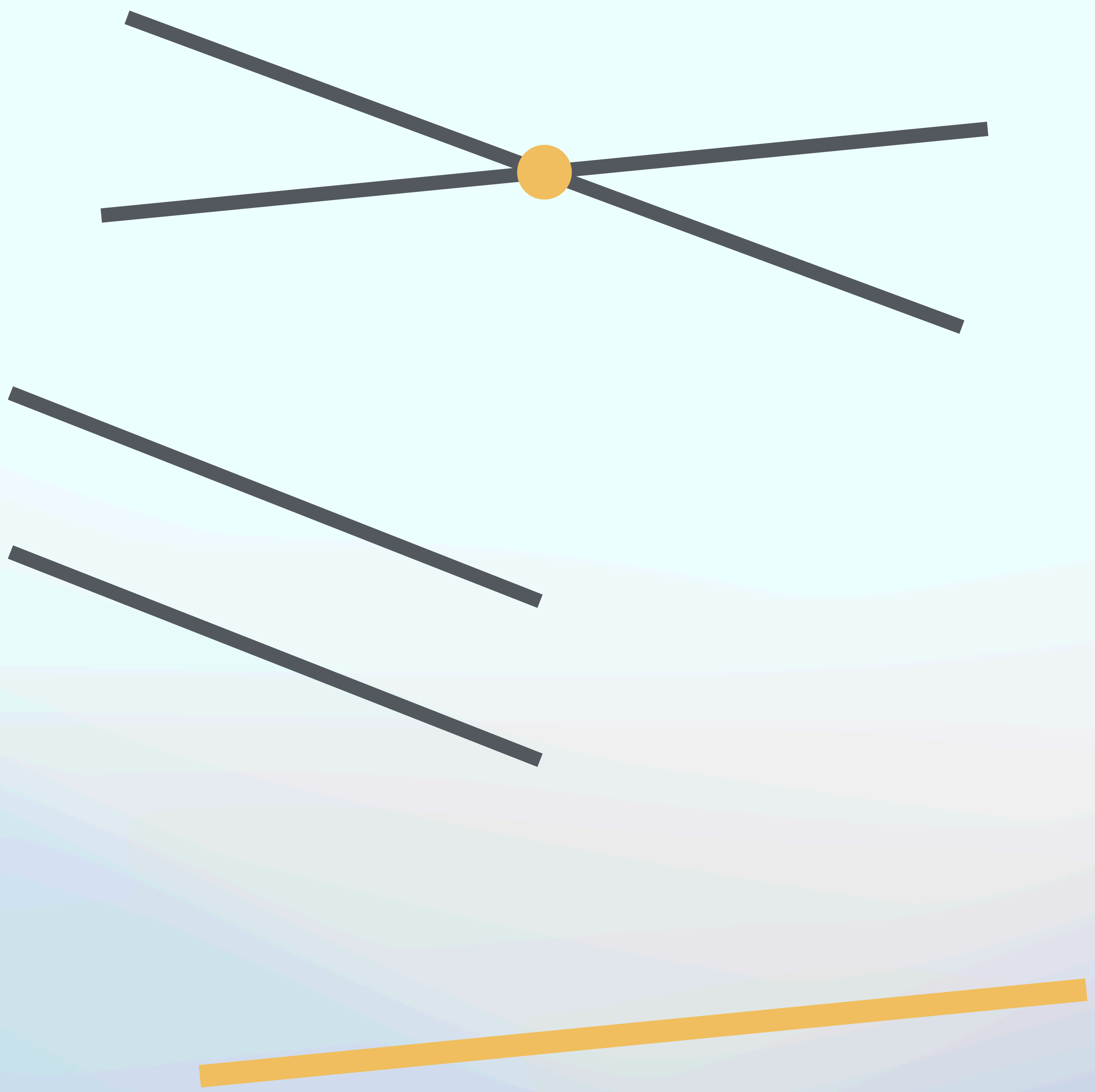


Infinite, if they're the same point

The notion of *generic*

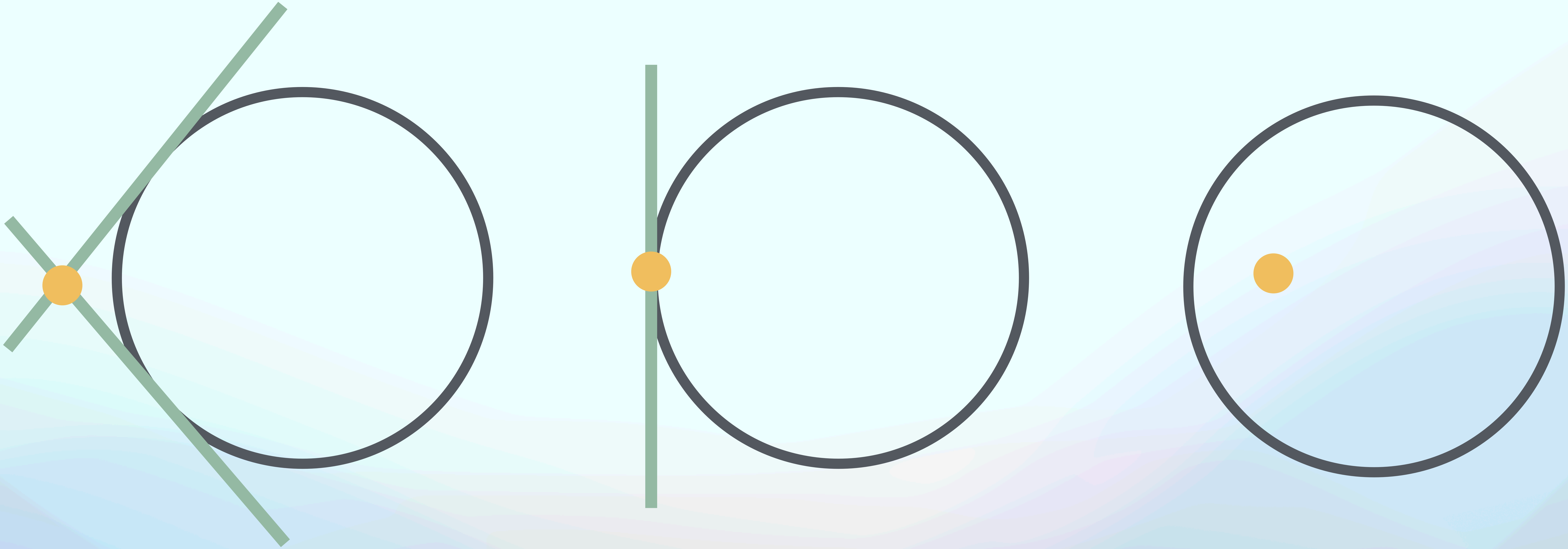
Something that is true *most of the time*

- Two (generic) lines intersect at a single point. Usually true, even though not always the case.



Circles and Points

Consider a Circle and a Point. How many lines tangent to the circle go through the point?



Two, if point outside

One, if point on circle

None, if point inside

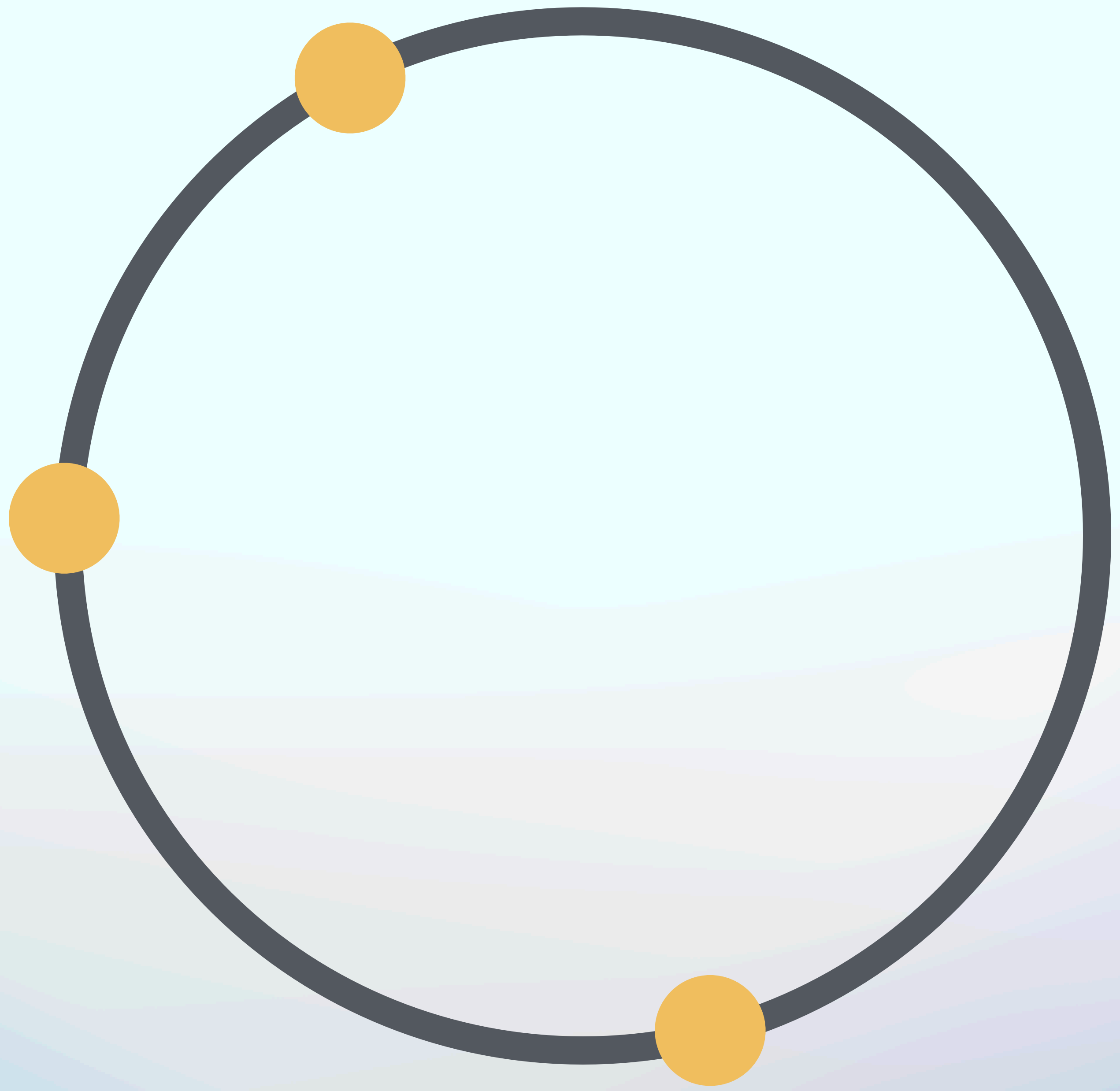
Problem of Apollonius

- Apollonius of Perga (c. 262 BC - c. 190 BC) posed and solved this problem himself.
- The question asks to count and construct circles on the plane satisfying a series of conditions involving points, lines and circles.

Problem of Apollonius

Case of three points

- Generically, how many circles are there through three given points?

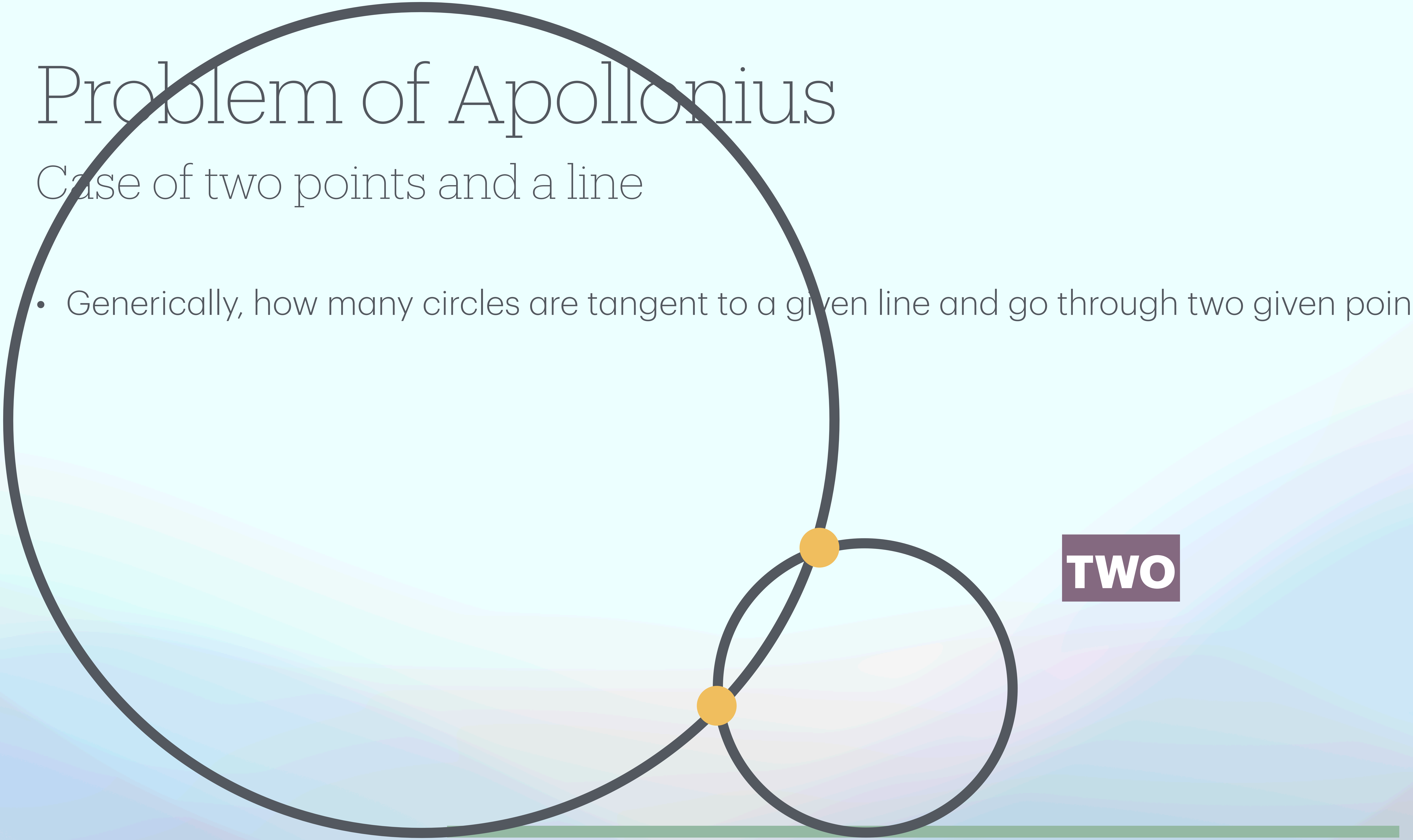


ONE

Problem of Apollonius

Case of two points and a line

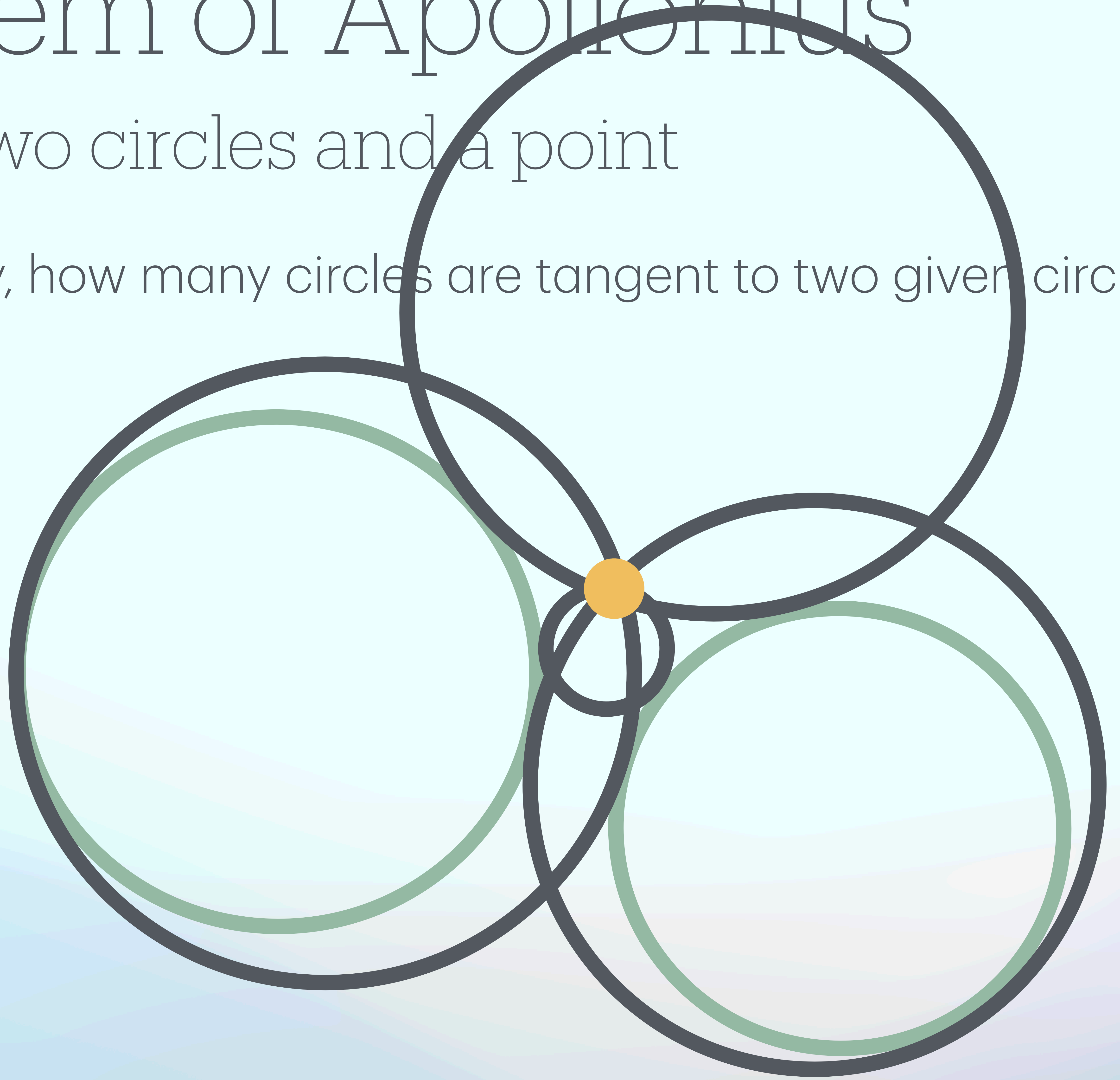
- Generically, how many circles are tangent to a given line and go through two given points?



Problem of Apollonius

Case of two circles and a point

- Generically, how many circles are tangent to two given circles and go through a given point?

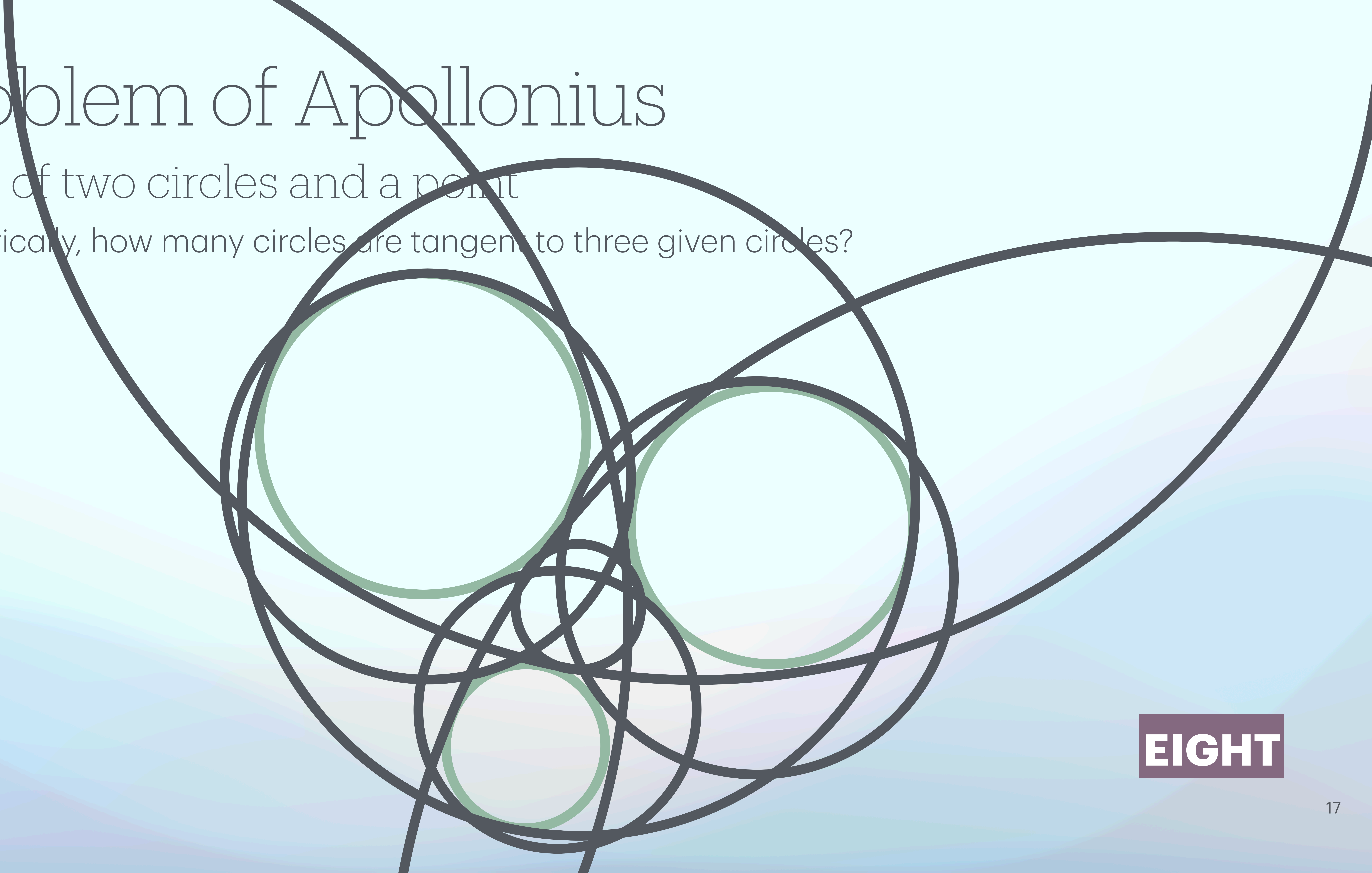


FOUR

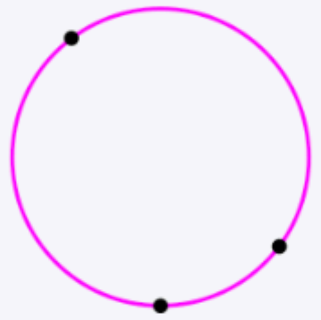
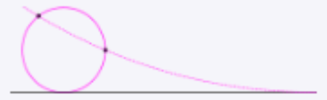
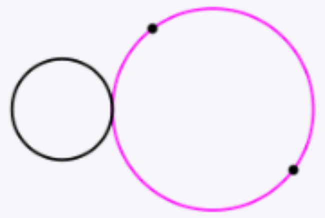
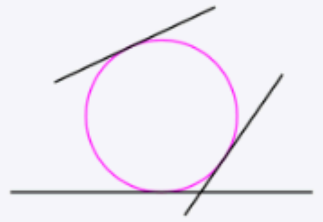
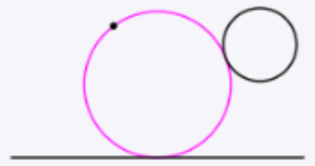
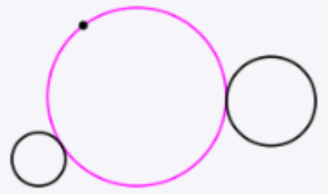
Problem of Apollonius

Case of two circles and a point

- Generically, how many circles are tangent to three given circles?



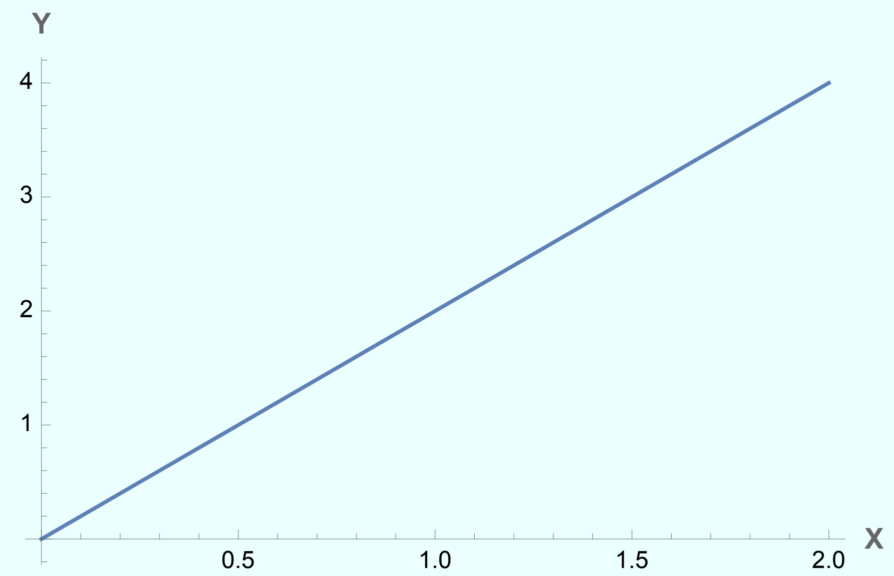
Problem of Apollonius

Index	Code	Given Elements	Number of solutions (in general)	Example (solution in pink; given objects in black)
1	**PPP**	three points	1	
2	**LPP**	one line and two points	2	
3	**LLP**	two lines and a point	2	
4	**CPP**	one circle and two points	2	
5	**LLL**	three lines	4	
6	**CLP**	one circle, one line, and a point	4	
7	**CCP**	two circles and a point	4	
8	**CLL**	one circle and two lines	8	
9	**CCL**	two circles and a line	8	
10	**CCC**	three circles (the classic problem)	8	

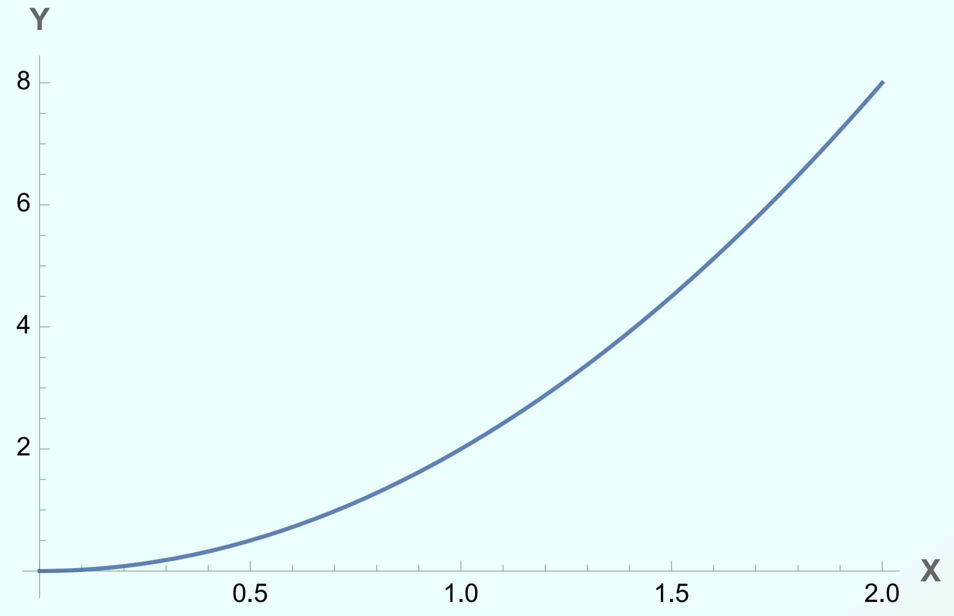
Fitting Lines in Curves

Algebra of Curves

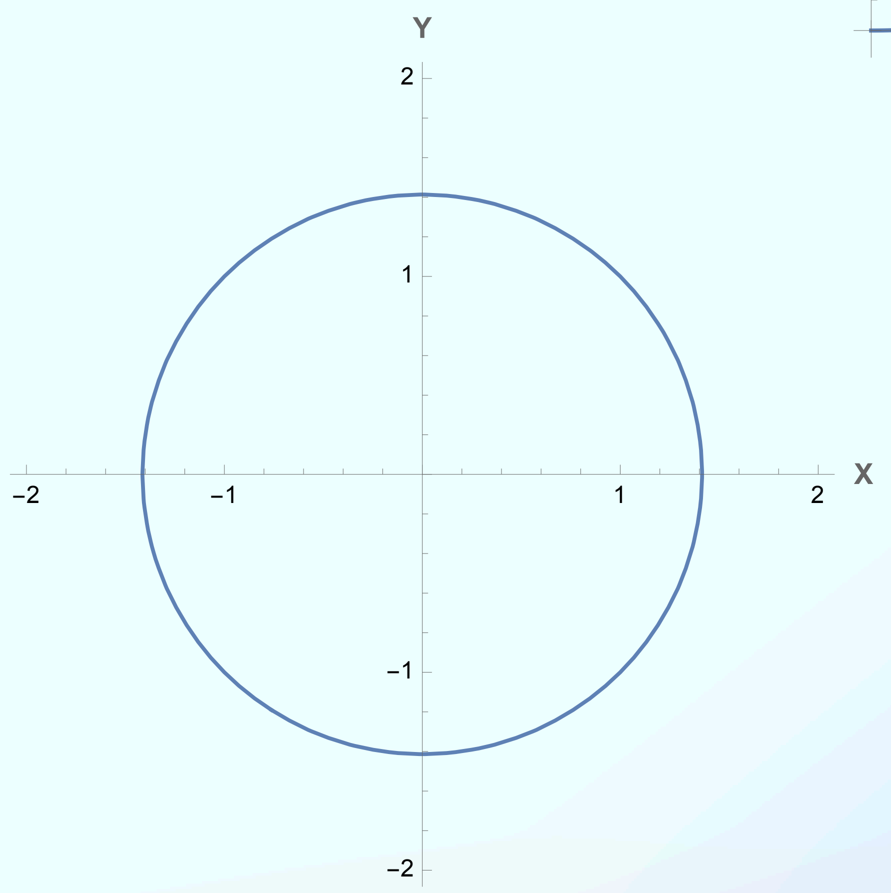
- This **linear** equation $Y = 2X$ defines a line in $\mathbb{R}^2$



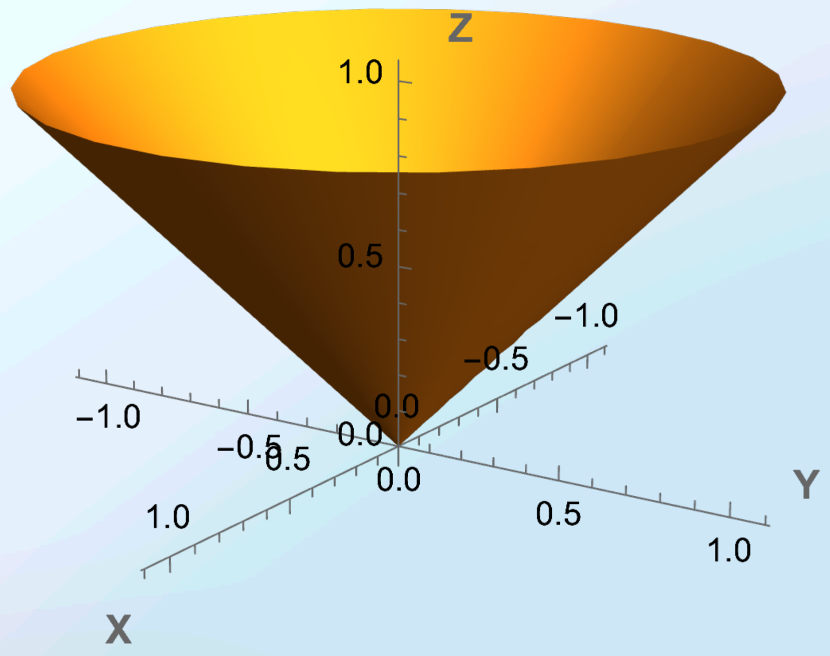
- This **quadratic** equation $Y = 2X^2$ defines a parabola in $\mathbb{R}^2$



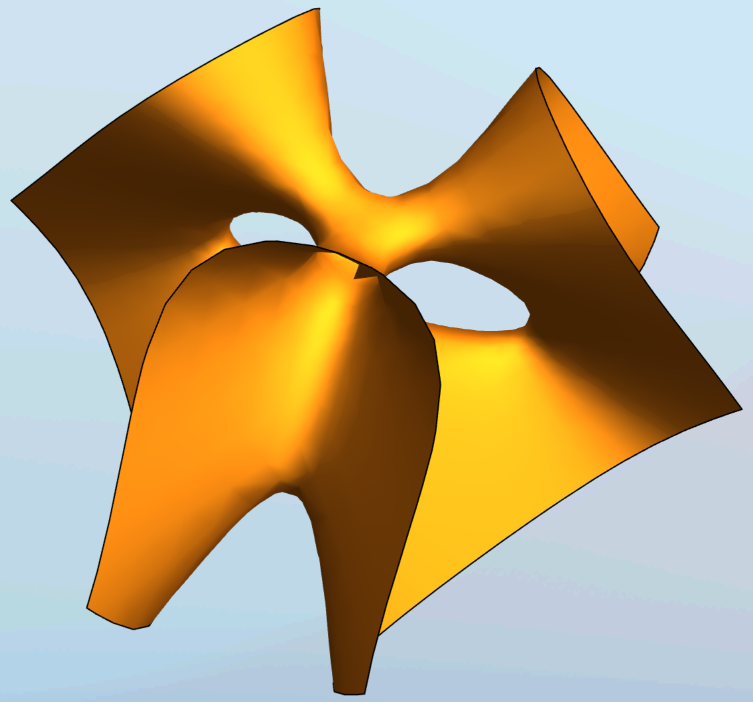
- This **quadratic** equation $2 = X^2 + Y^2$ defines a circle in $\mathbb{R}^2$



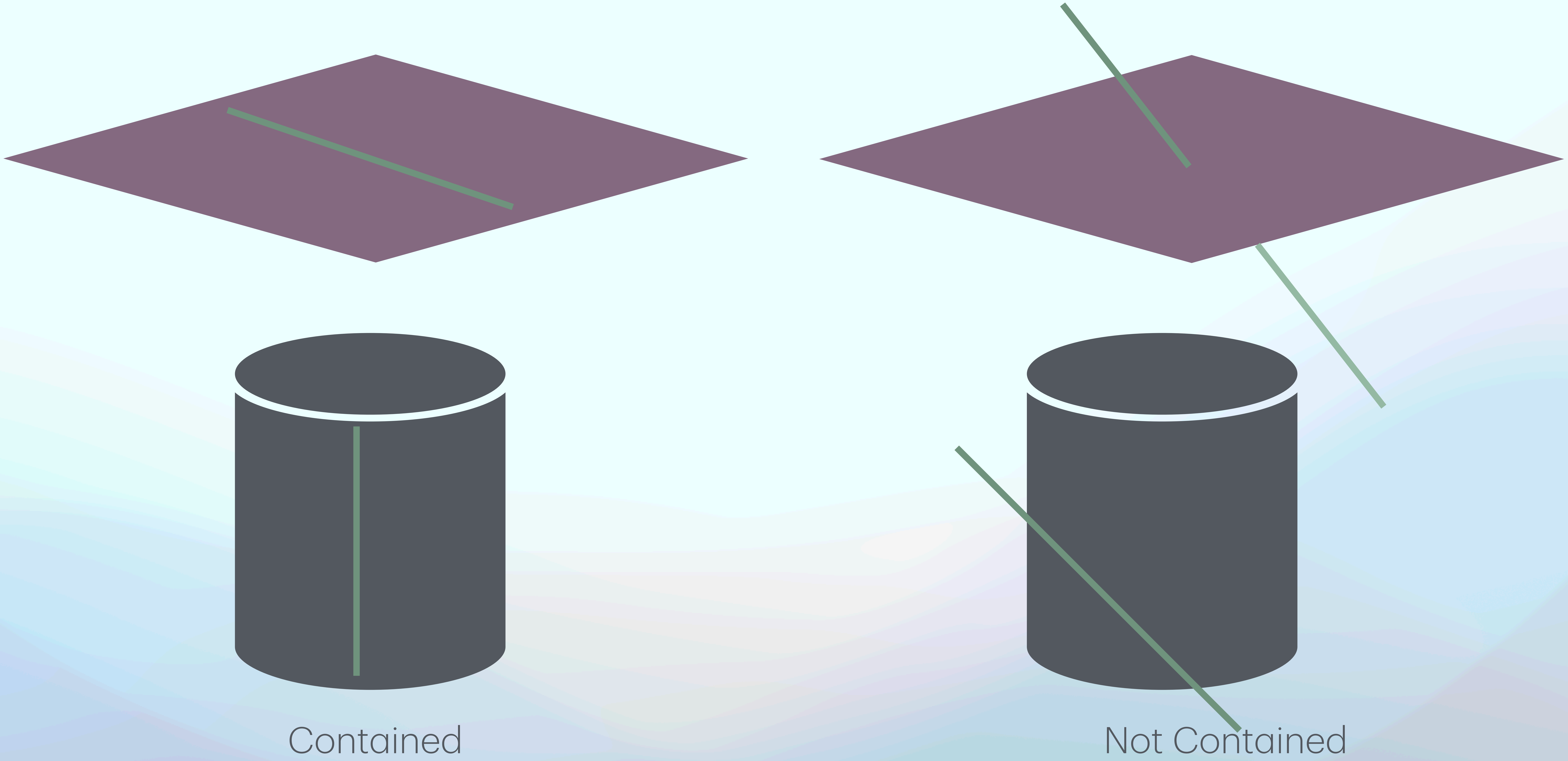
- This **quadratic** equation $Z^2 = X^2 + Y^2$ defines a cone in $\mathbb{R}^3$



- This **cubic** equation $X^3 + Y^3 + Z^3 + 1 = (X + Y + Z + 1)^3$ defines a surface in $\mathbb{R}^3$



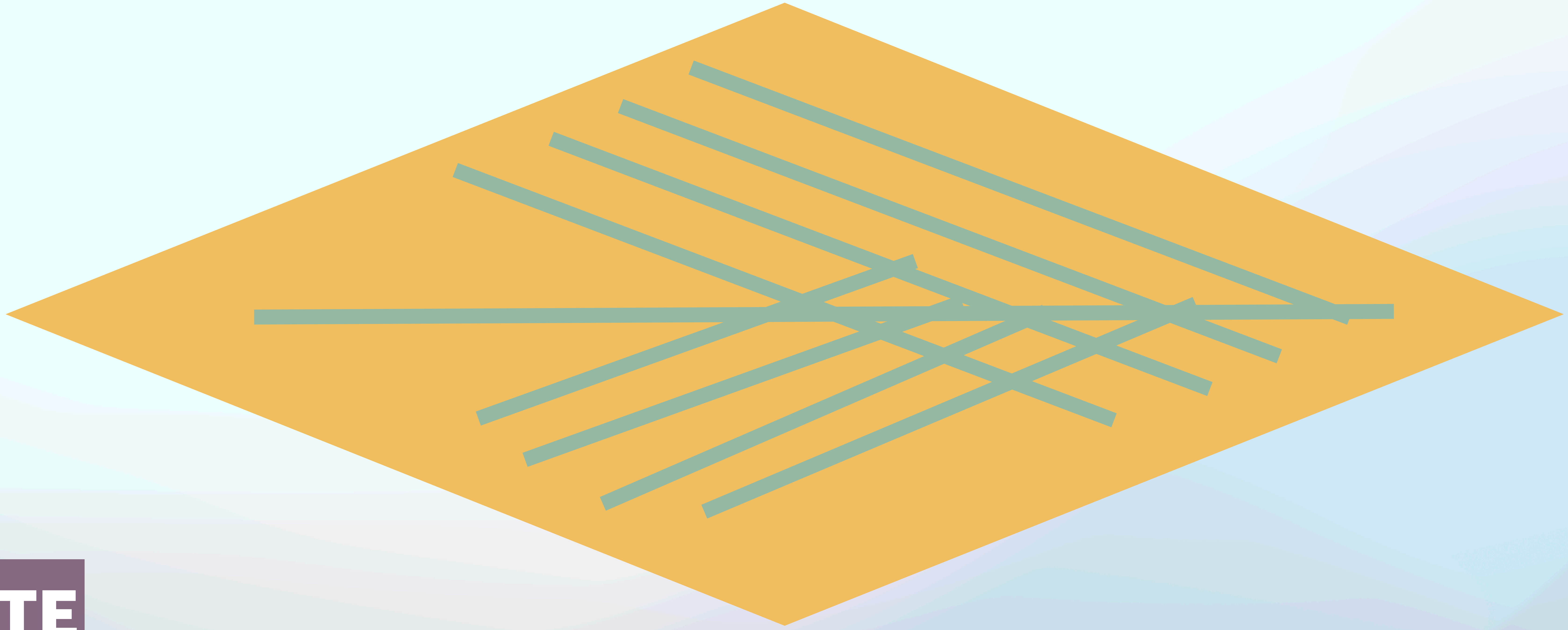
How many lines can you fit inside a surface?



Counting Lines

How many lines are contained in a linear surface (a plane)?

$X - Y = Z$

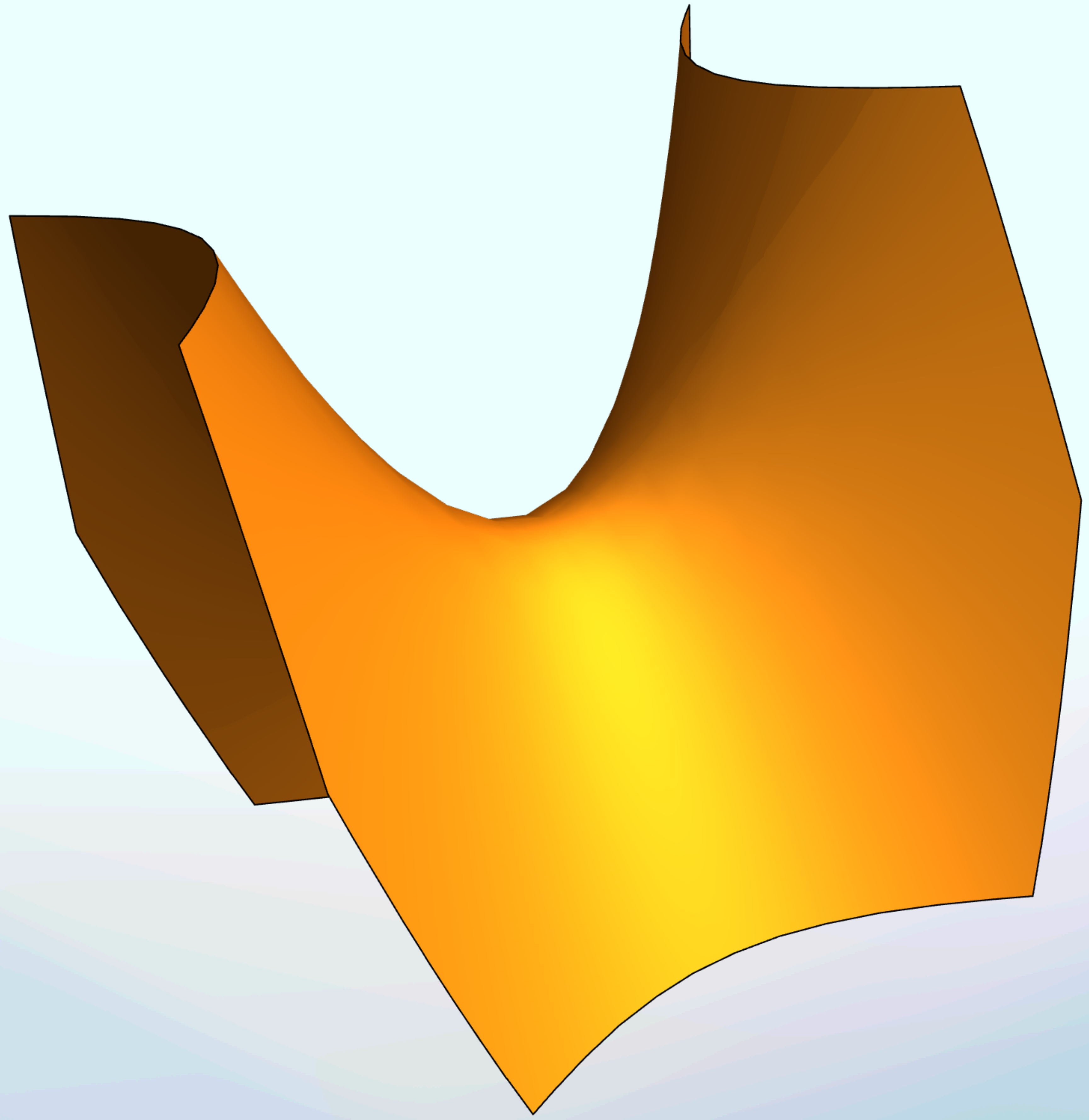


INFINITE

Counting Lines

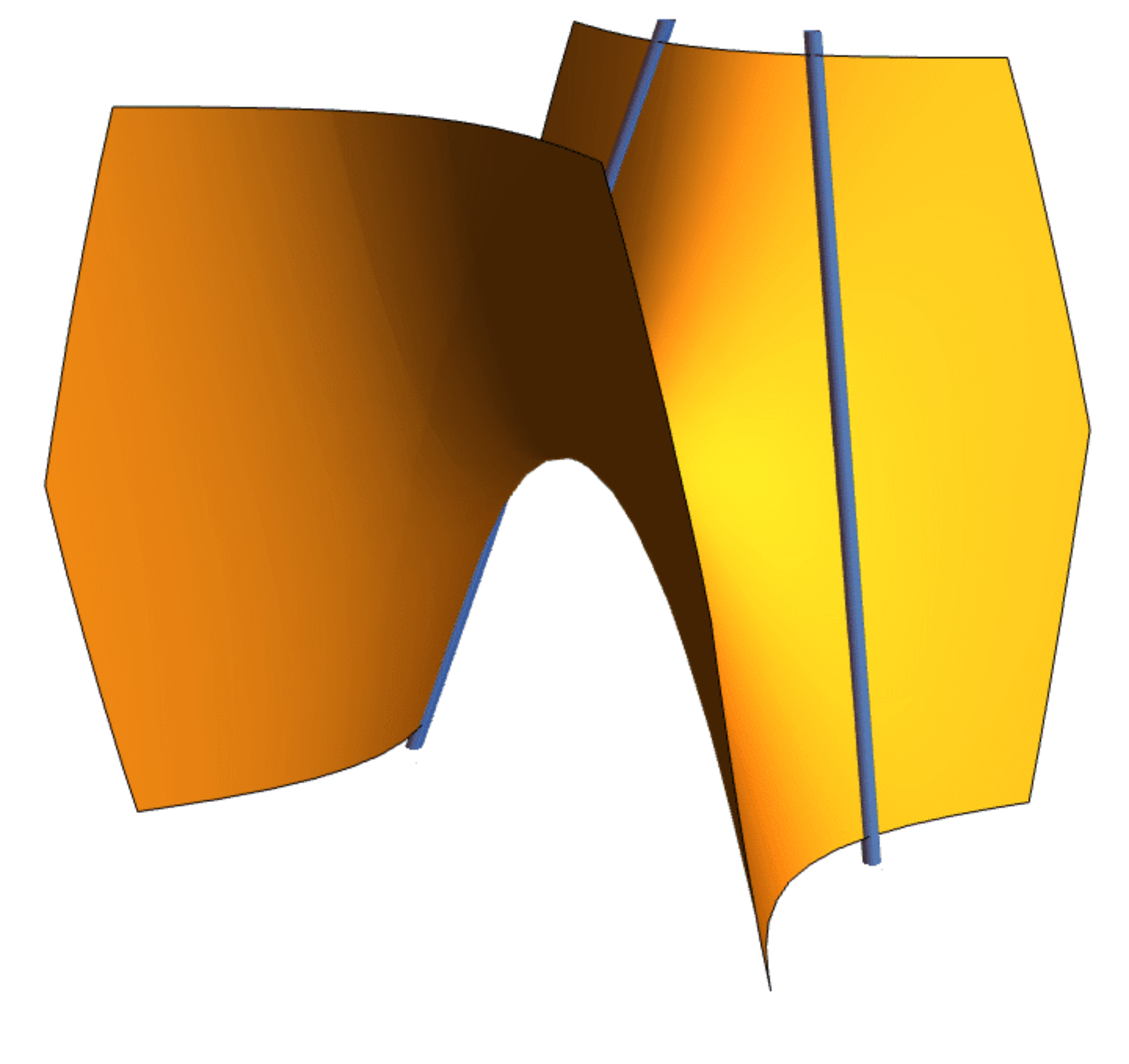
How many lines are contained in a quadratic surface?

$$X^2 - Y^2 = Z^2$$



Counting Lines

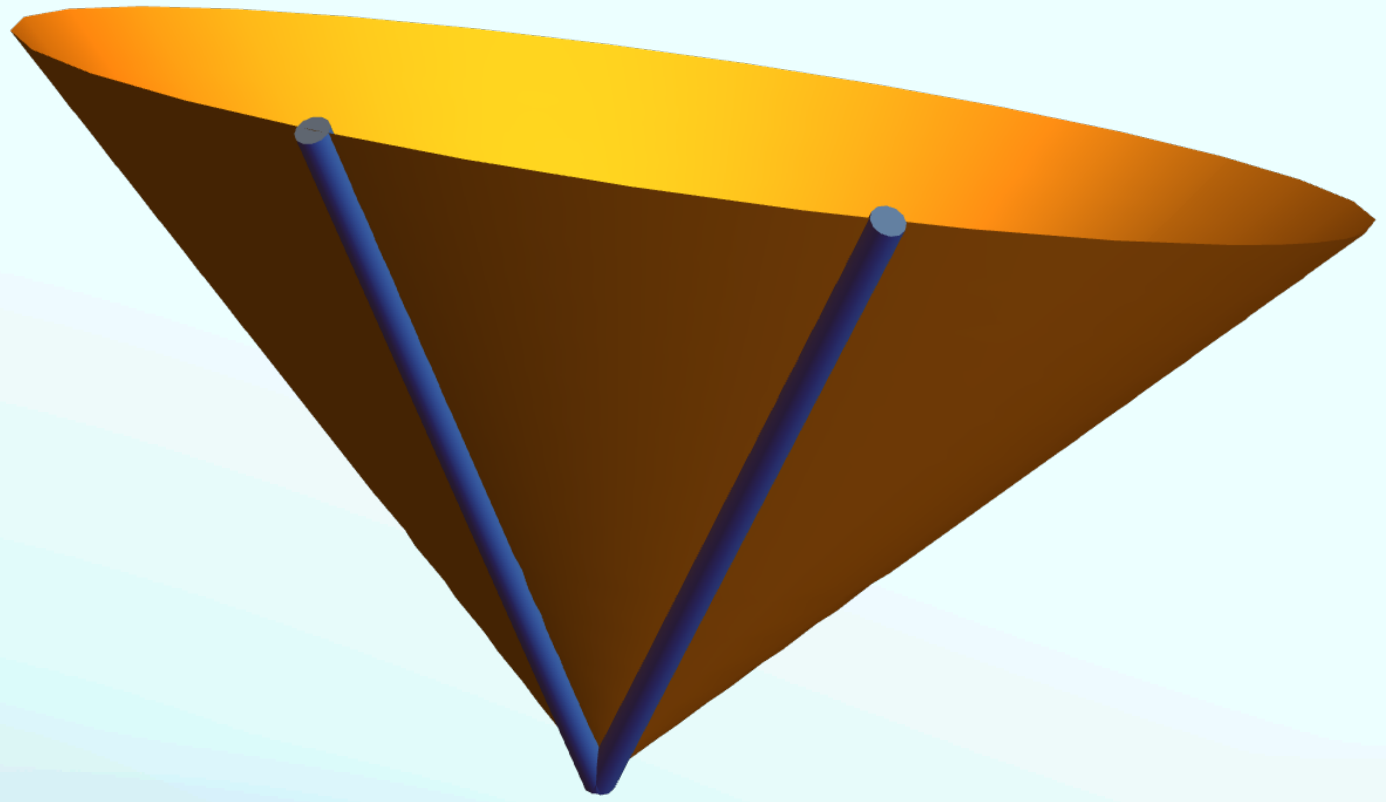
How many lines are contained in a quadric surface?



INFINITE

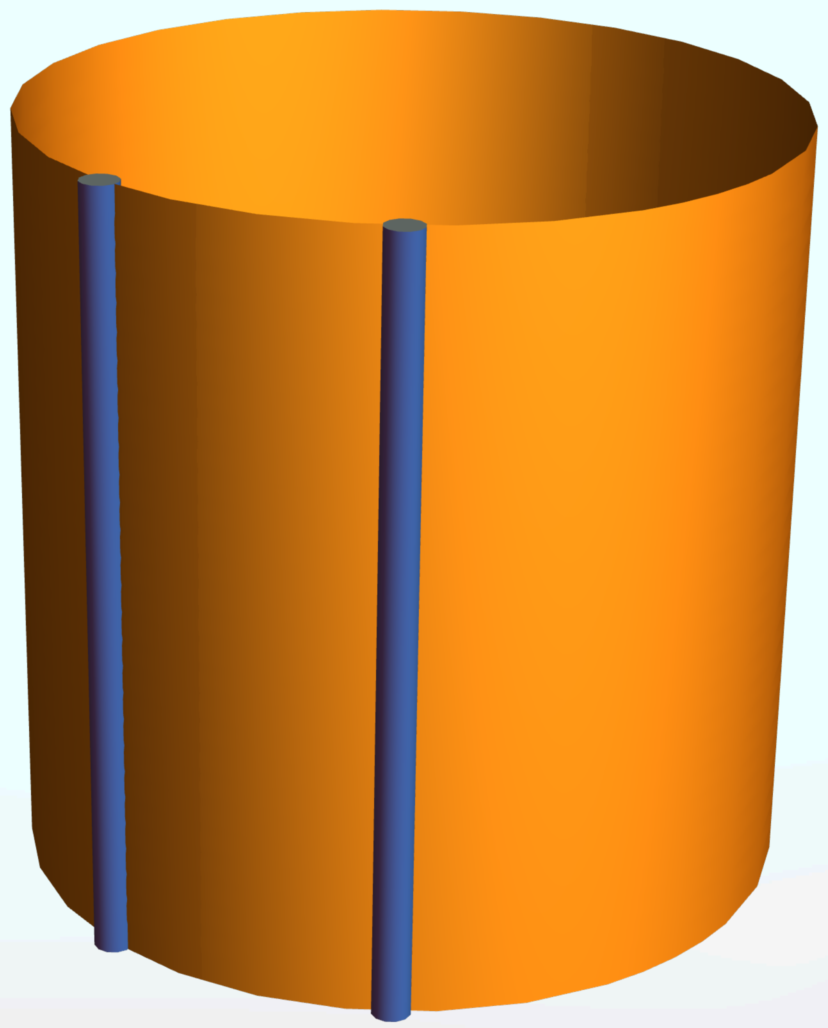
Ruled Surfaces

All these quadratic surfaces have infinite lines in them!



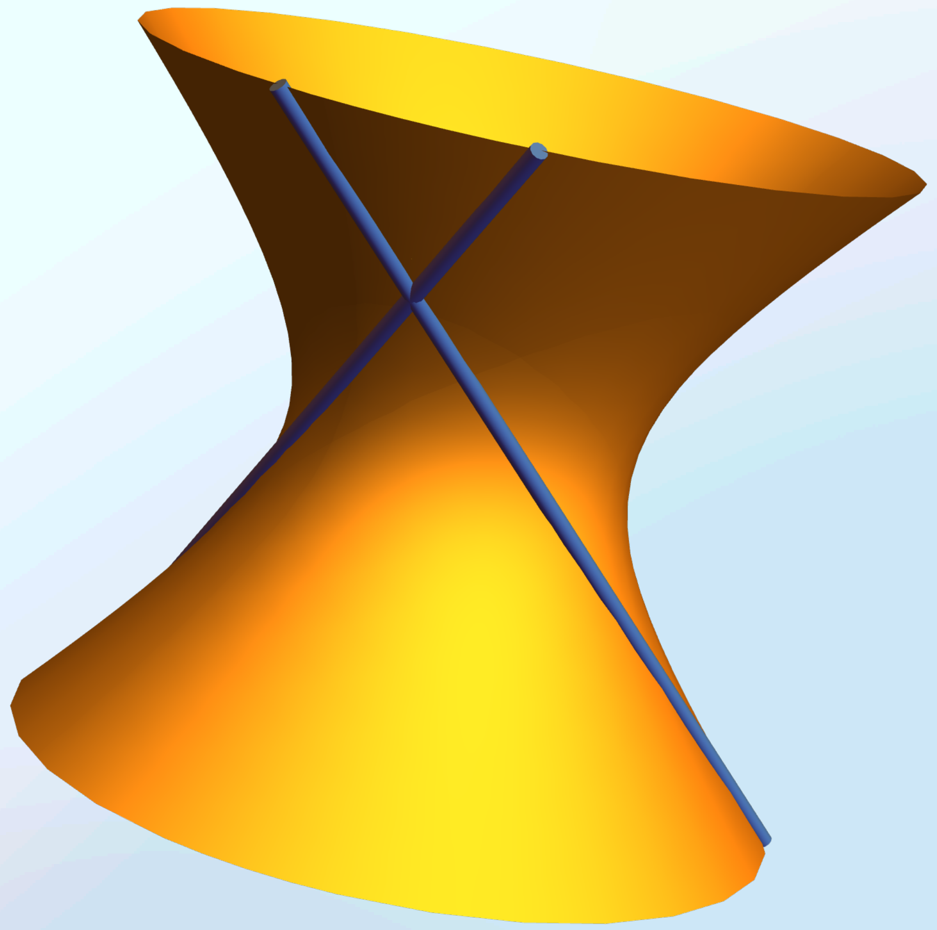
$X^2 + Y^2 = Z^2$

Cone



$X^2 + Y^2 = 1$

Cylinder



$X^2 + Y^2 = Z^2 + 1$

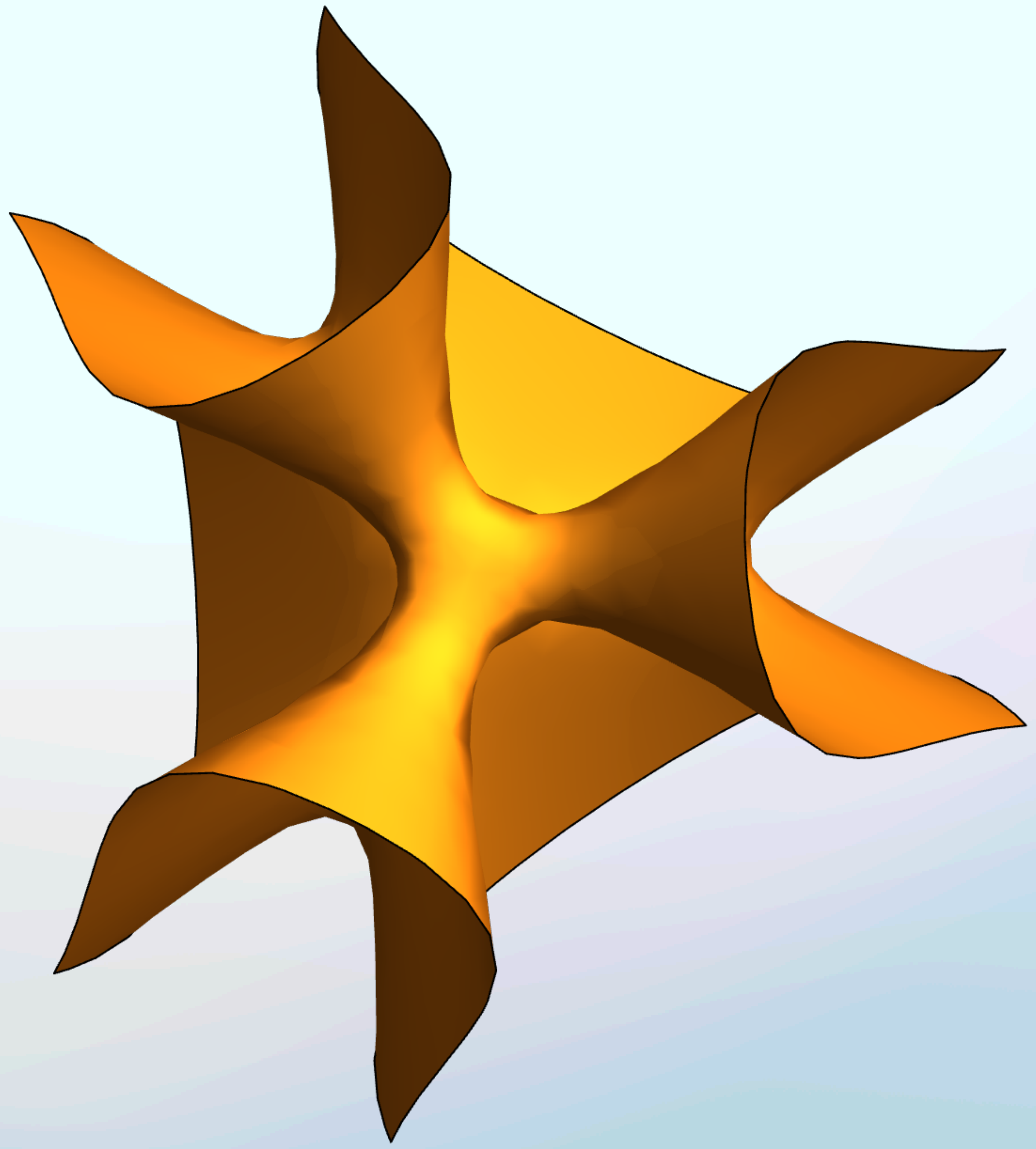
Hyperboloid

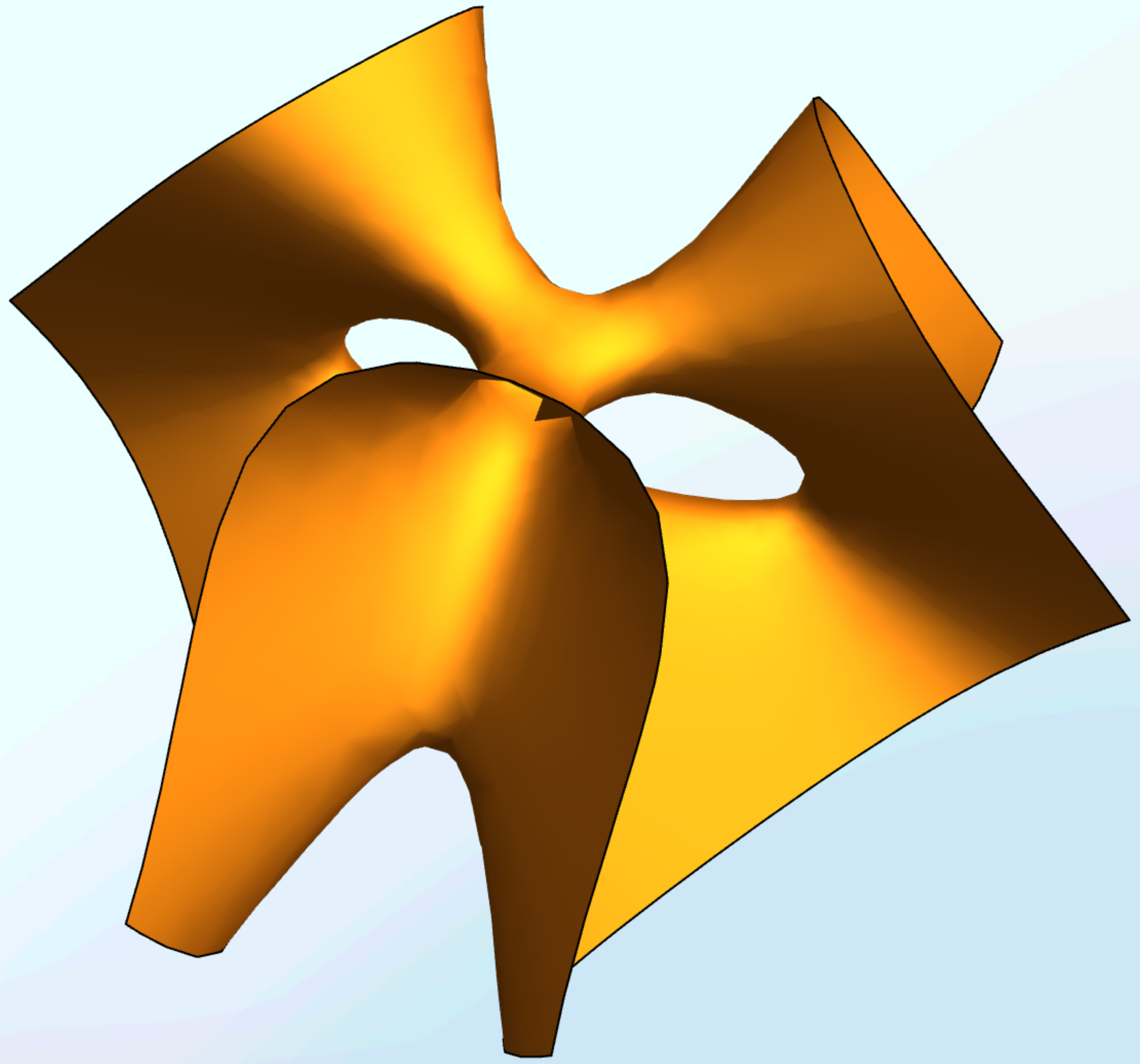
Counting Lines

How many lines go through a *cubic* surface?

$$X^3 + Y^3 + Z^3 + 1 = (X + Y + Z + 1)^3$$

TWENTY-SEVEN





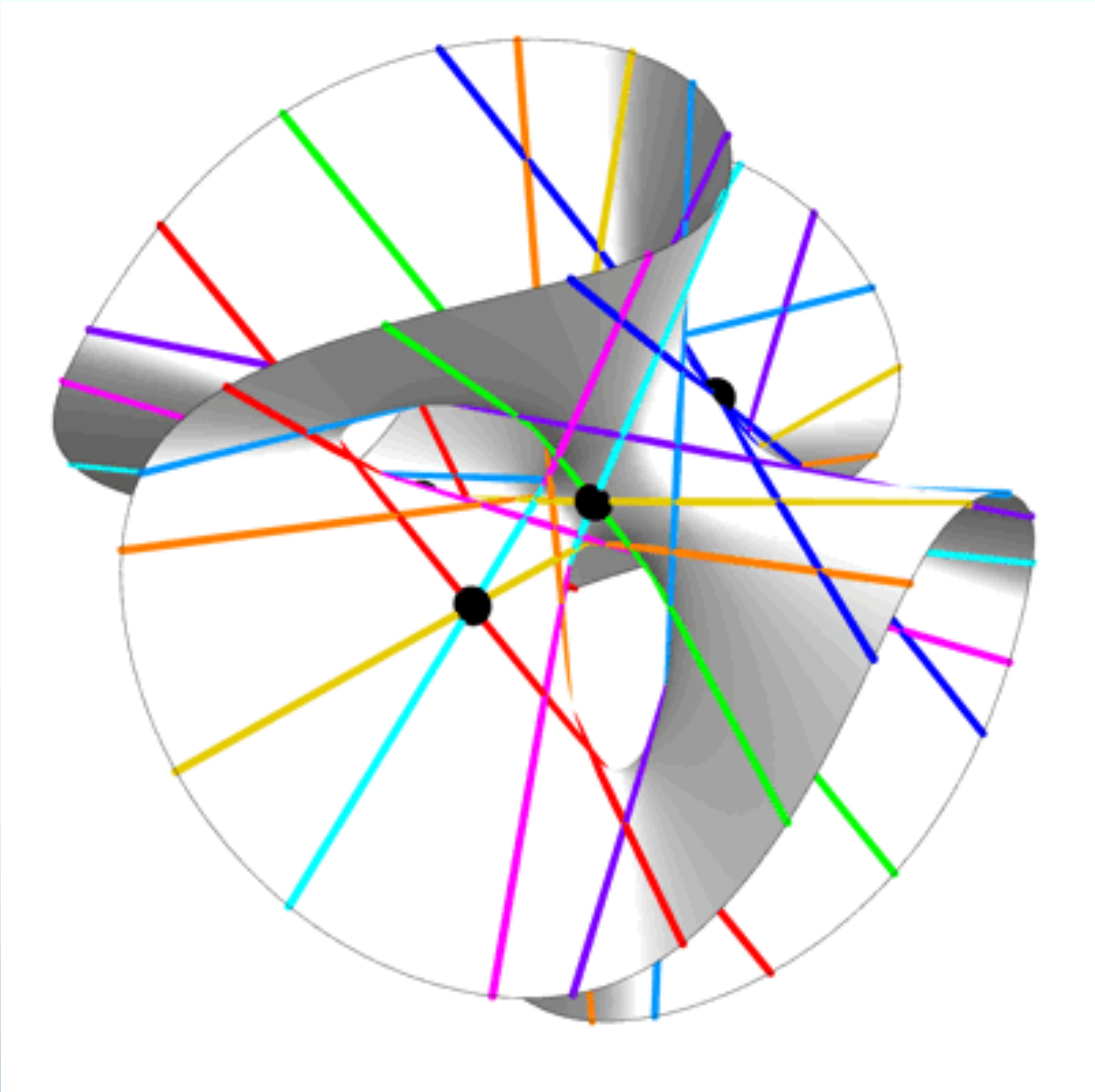
Clebsch Surface

Counting Lines

How many lines go through a cubic surface?

$$X^3 + Y^3 + Z^3 + 1 = (X + Y + Z + 1)^3$$

TWENTY-SEVEN

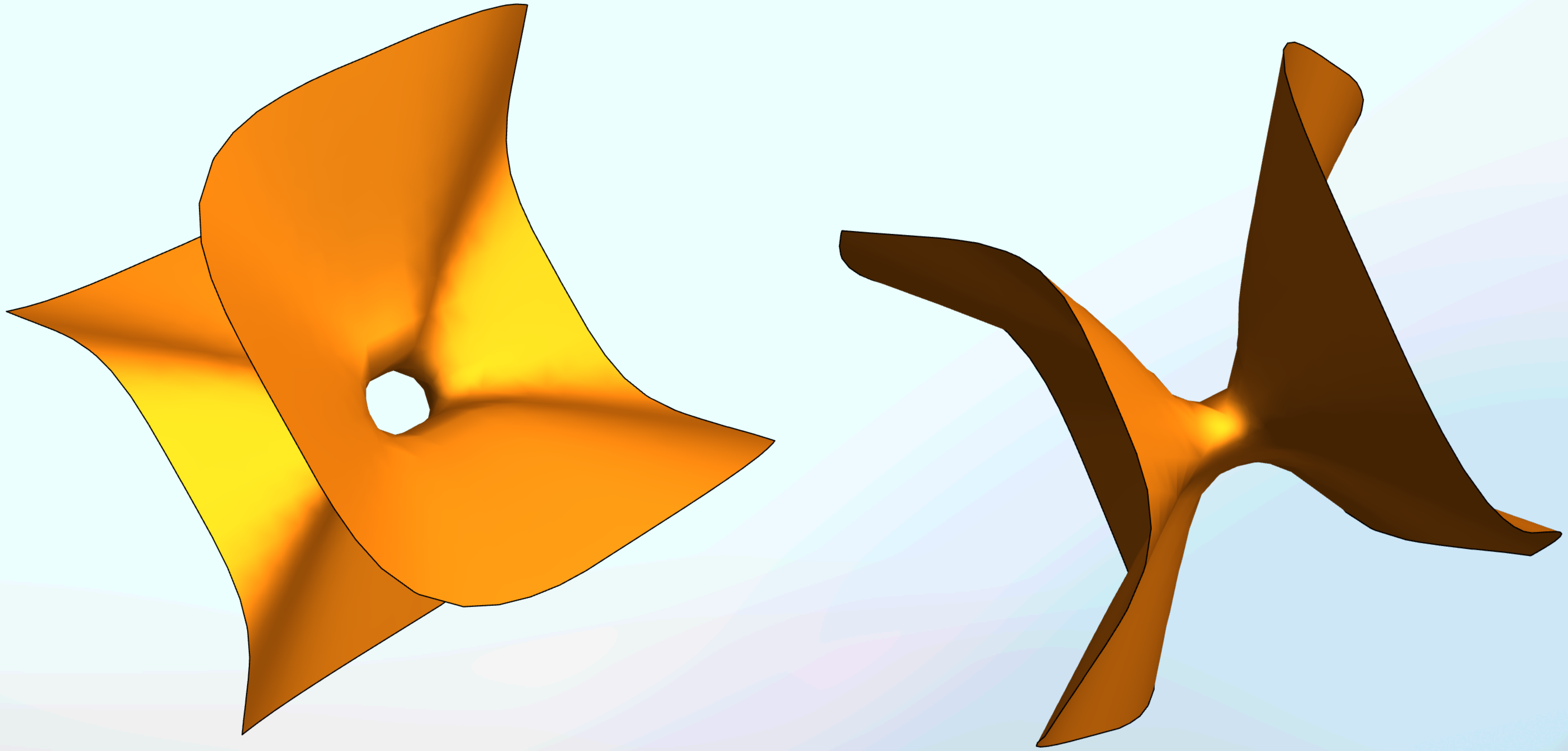


Counting Lines

How many lines go through a quartic surface?

$$X^4 - XY^3 = Z^4 = Z$$

FIFTY-SIX



Schur Surface

Counting Lines

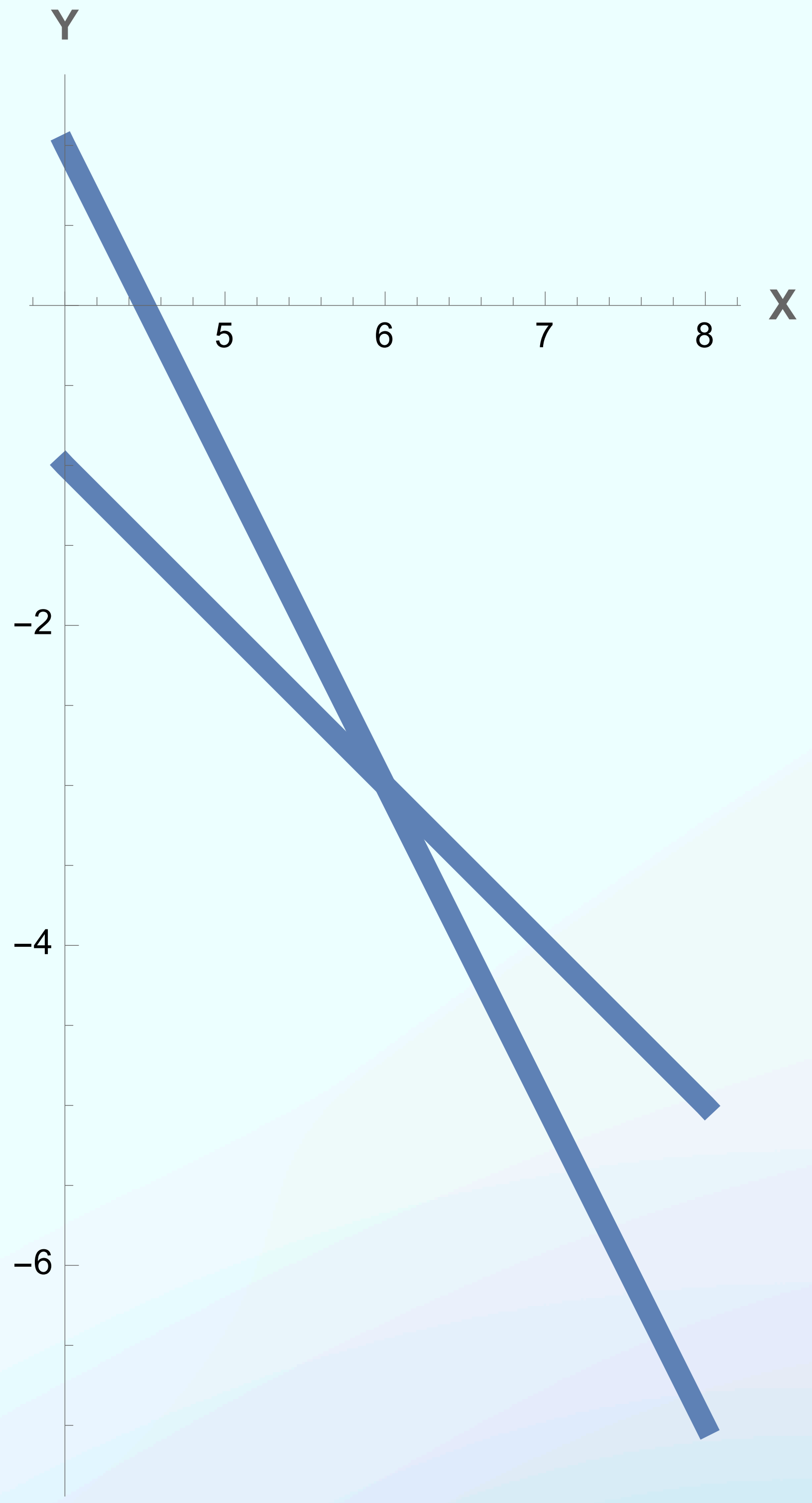
How many lines are contained in a quadratic surface?

Degree	Maximum Number of Real Lines	Maximum Number of Lines	Name
1	infinity	infinity	Plane
2	infinity	infinity	Quadratic
3	27	27	Cubic
4	56	64	Quartic
5	??	??	Quintic

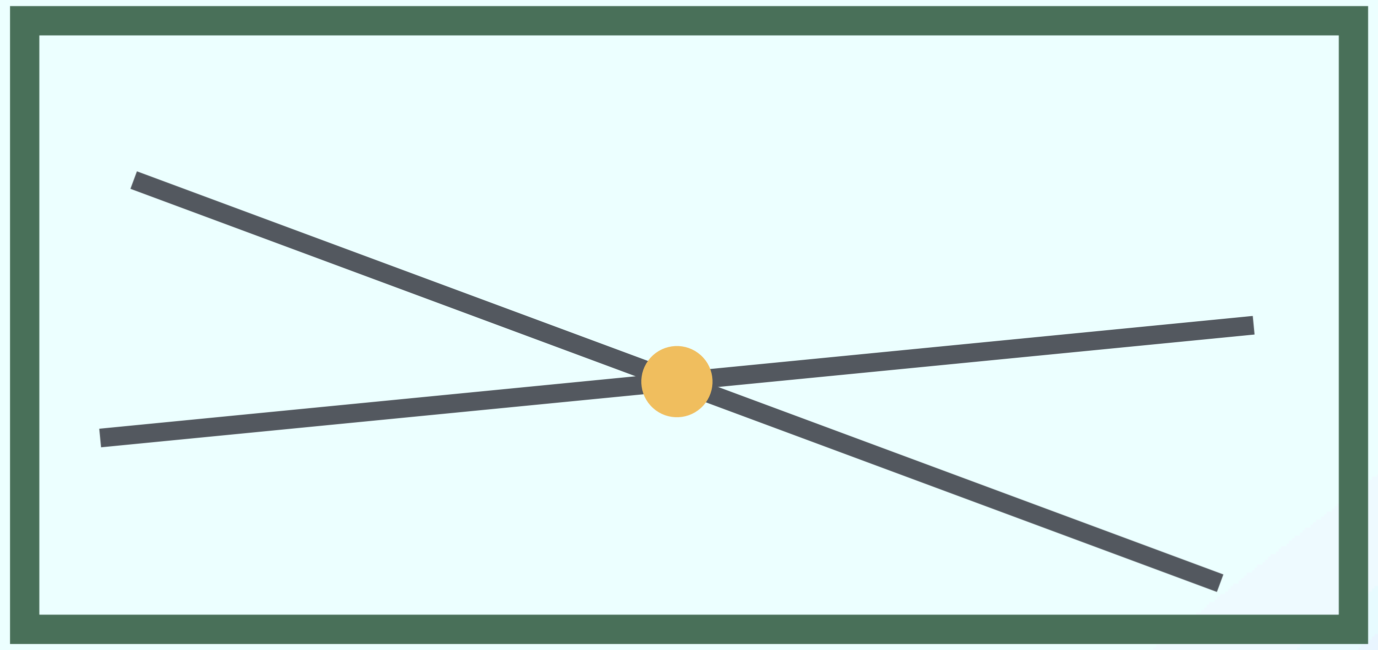
Intersection Theory

Solving Systems of Equations

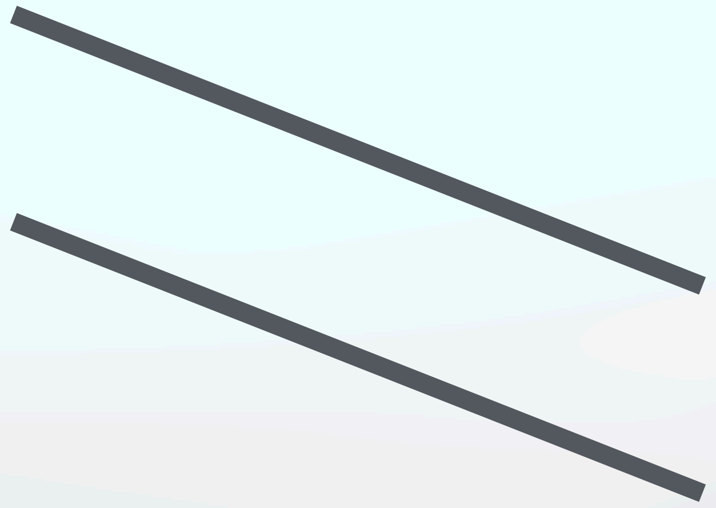
How many solutions does the system $\begin{cases} 2x + y = 9 \\ x + y = 3 \end{cases}$ have?



- Generically, this is the intersection of two lines in $\mathbb{R}^2$:
  - Intersecting: **one** solution



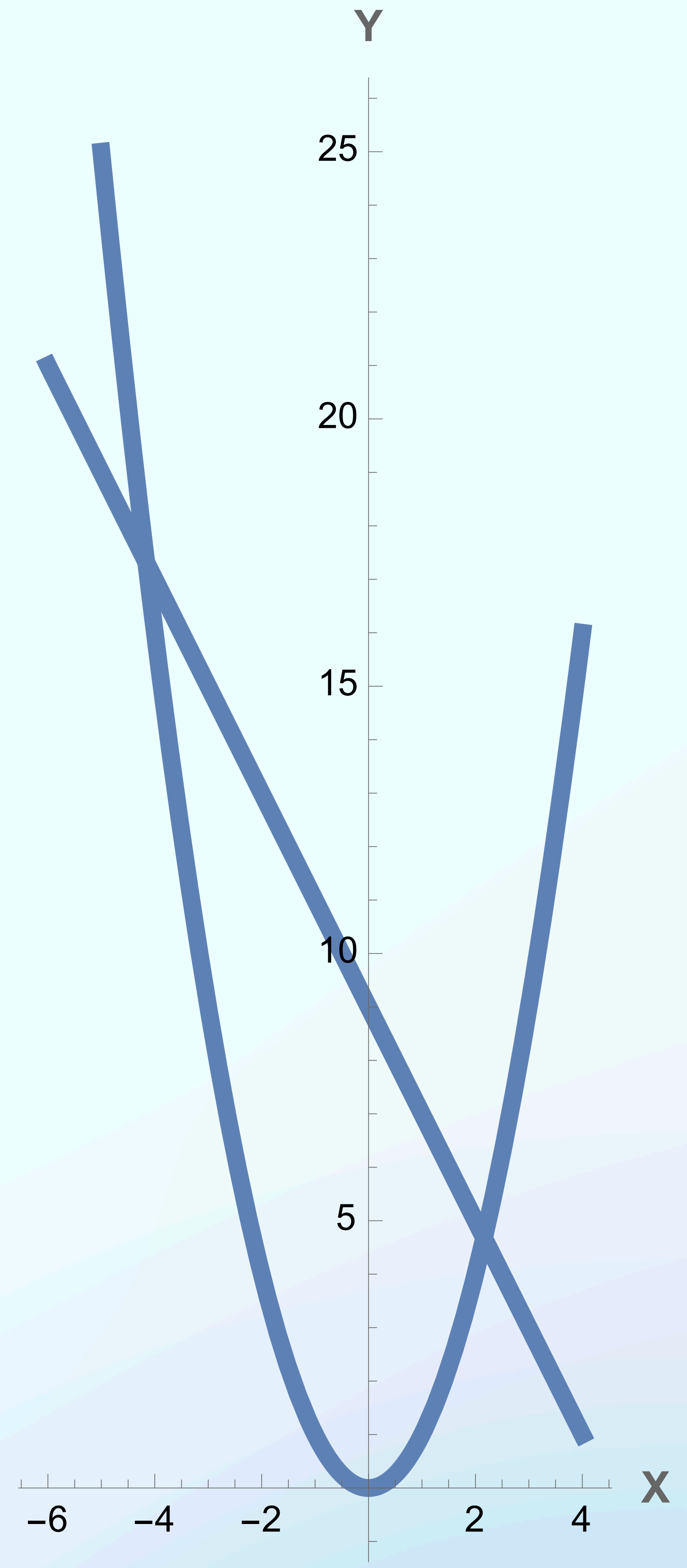
  - Parallel: **zero** solutions



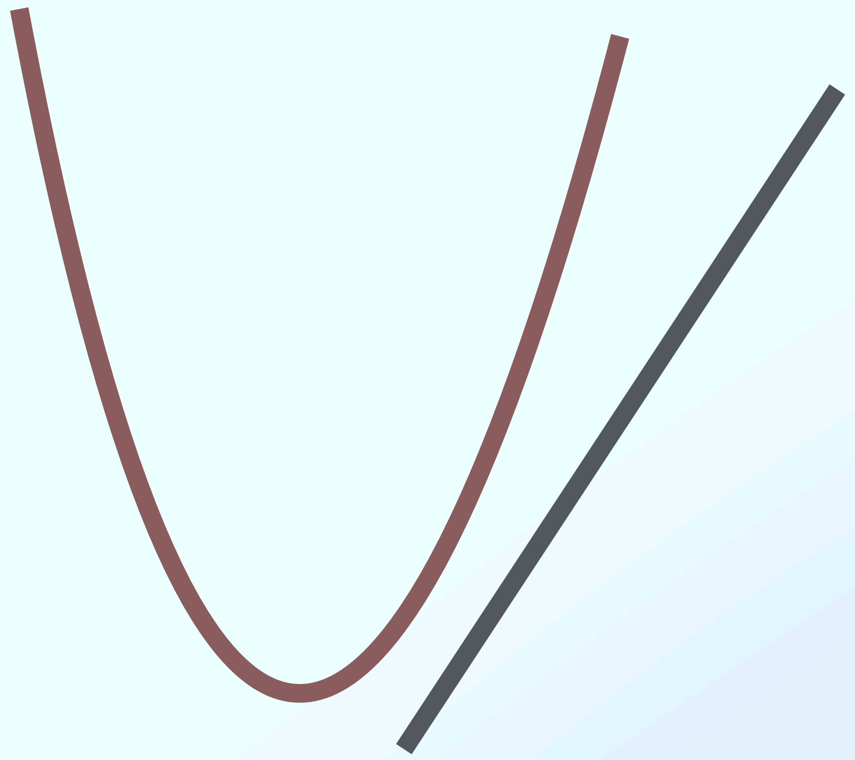
  - Coincident: **infinite** solutions

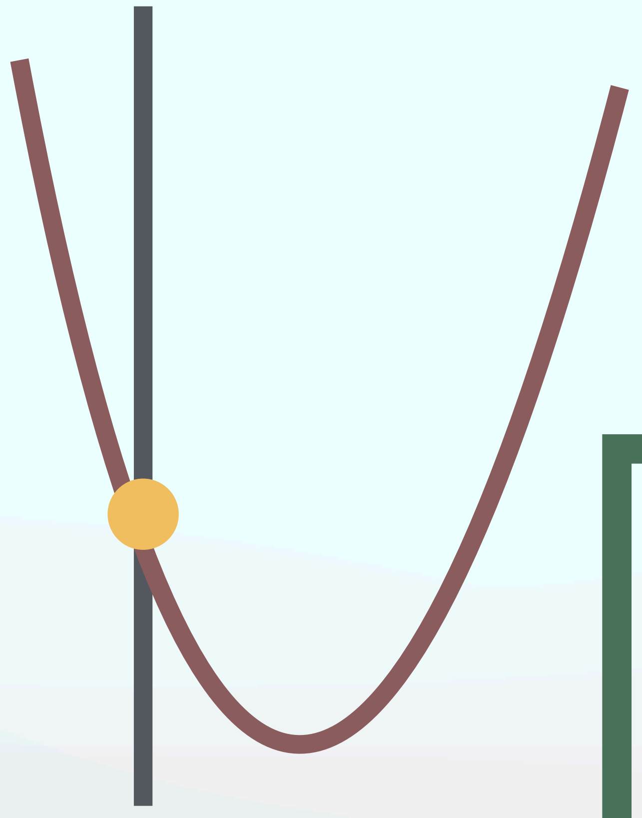
Solving Systems of Equations

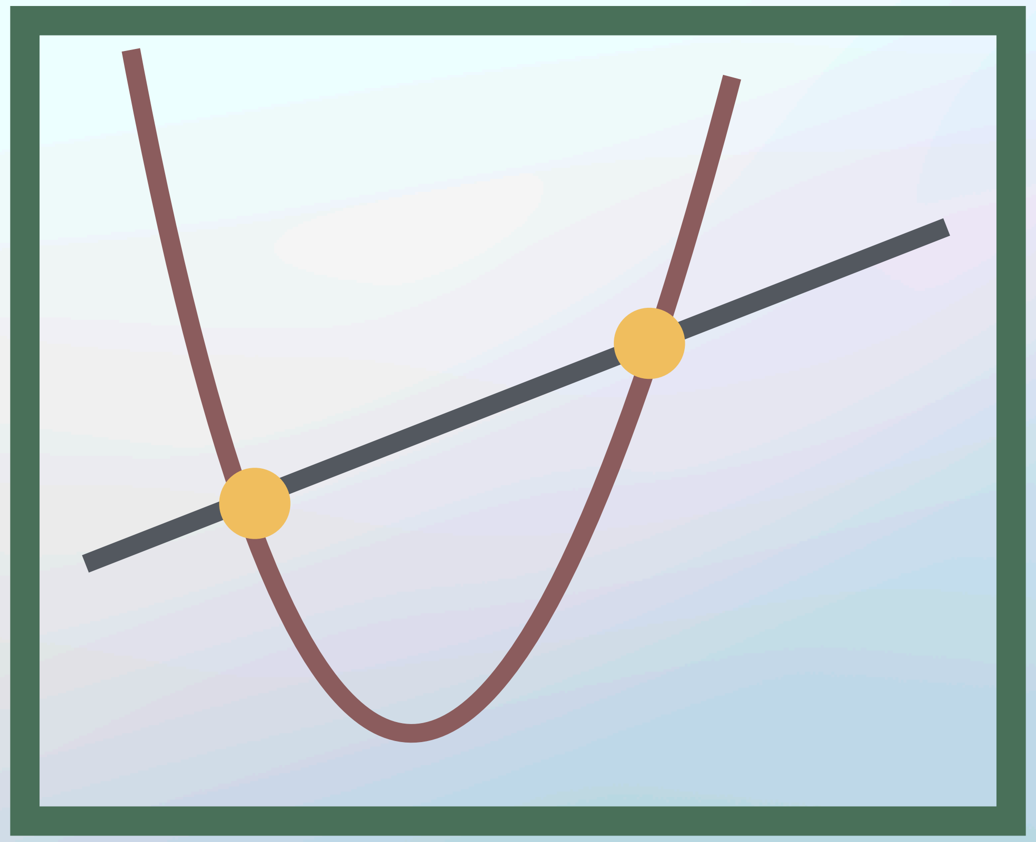
How many solutions does the system $\begin{cases} 2x + y = 9 \\ y = x^2 \end{cases}$ have?



- Generically, this is the intersection of a line and a parabola in $\mathbb{R}^2$:
  - Disjoint: **zero** solutions
  - Intersecting: **one** solution
  - Intersecting: **two** solutions

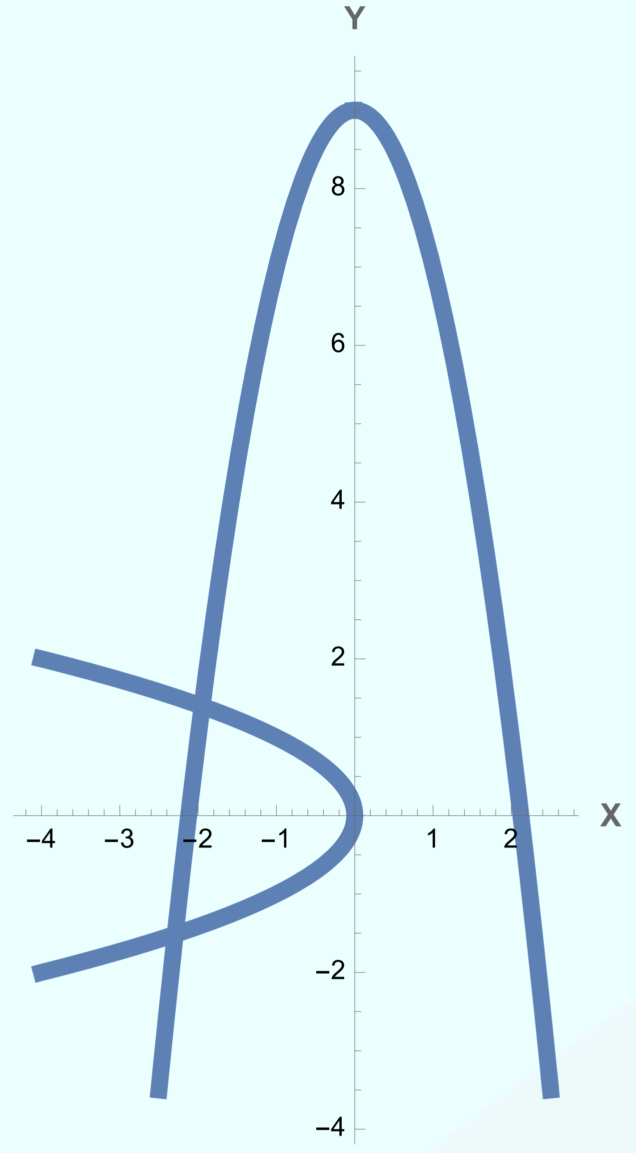




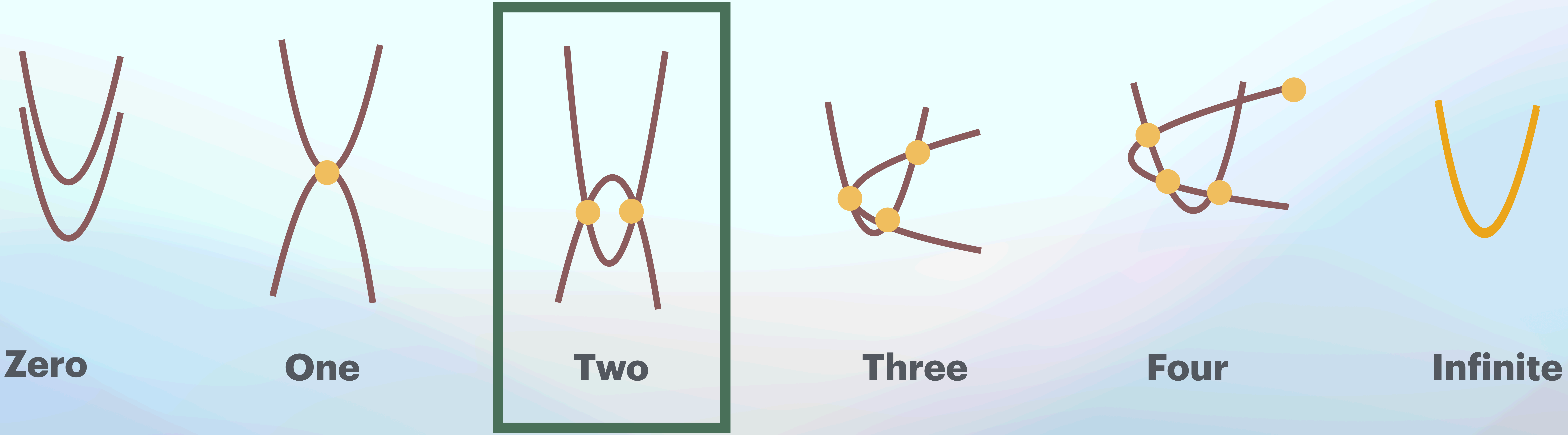


Solving Systems of Equations

How many solutions does the system $\begin{cases} 2x^2 + y = 9 \\ y^2 = -x \end{cases}$ have?



- Generically, this is the intersection of two parabolas in $\mathbb{R}^2$:



Counting Intersections

suppose we have n polynomials of degrees $d_1, d_2, \ldots, d_n$ in $\mathbb{R}^n$ then either

- The number of intersection points is infinite or…
- The number of intersection points is, generically, **at most** the product of the quantities

$$d_1 \times d_2 \times \cdots \times d_n$$

- In $\mathbb{R}^2$

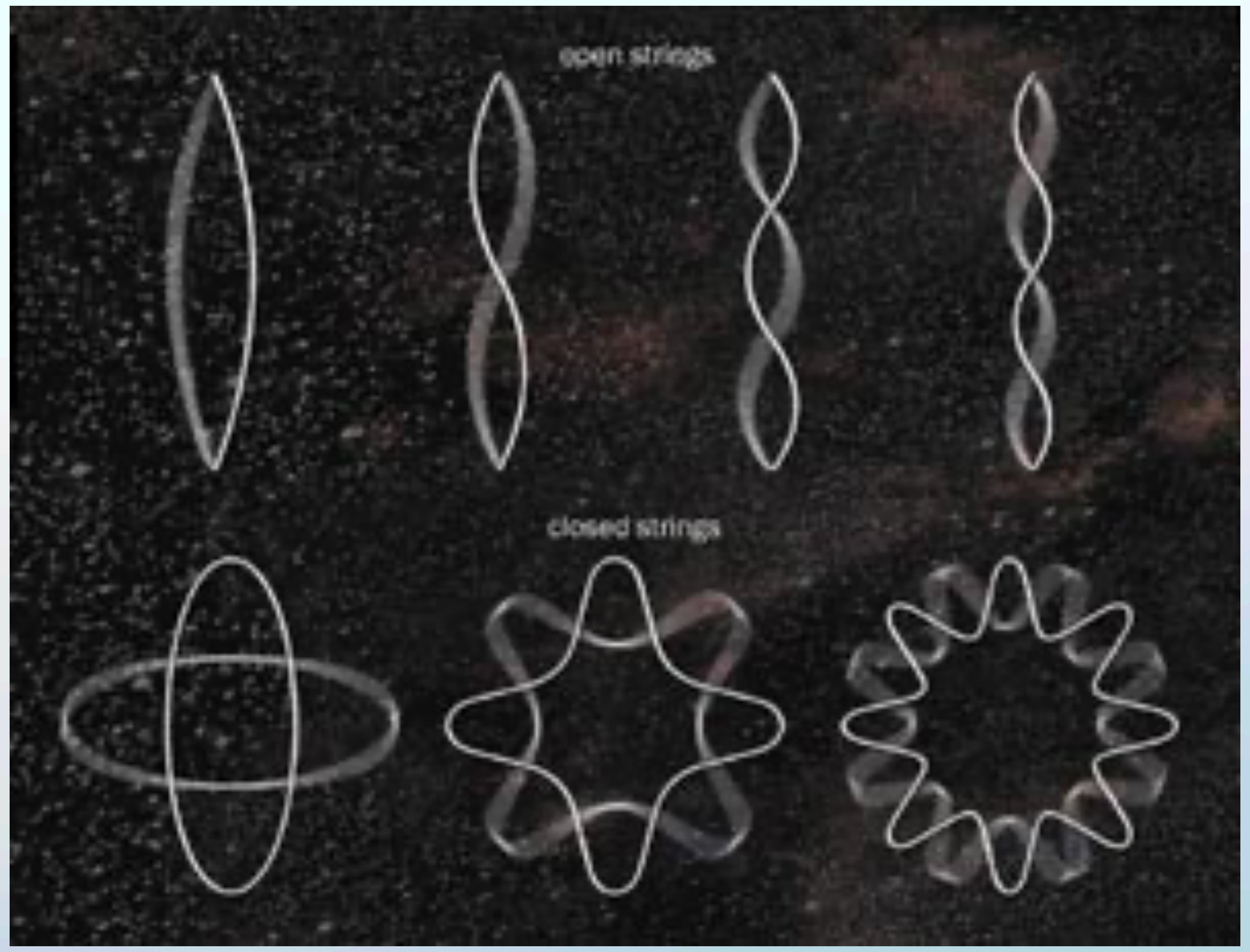
d1	**d2**	Max intersections
1	1	1
1	2	2
2	2	4
3	7	21

Graphs Sums in String Theory

String Theory

What is it about?

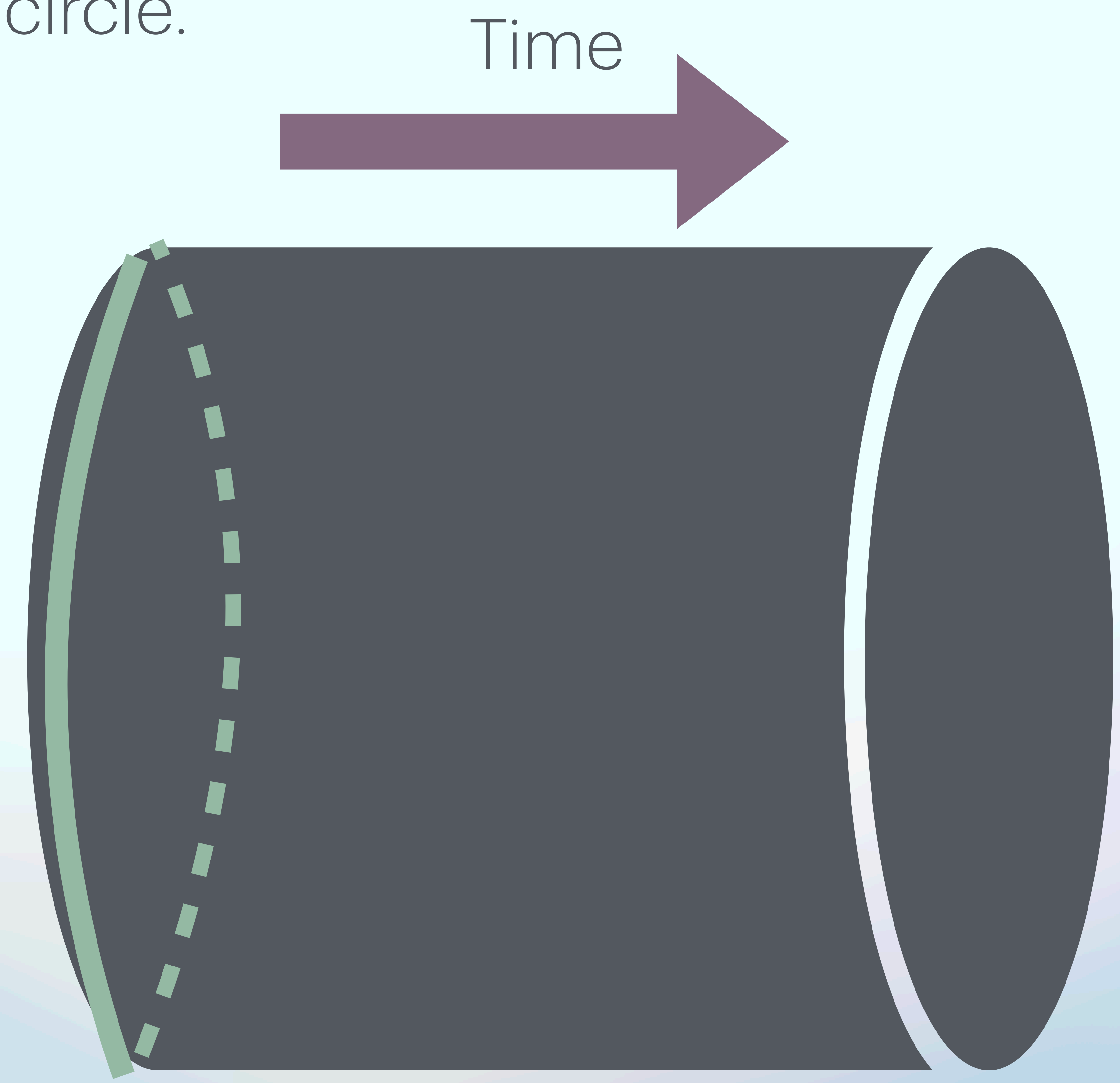
- String Theory is a theory in physics that suggests the basic building blocks of the universe are not tiny particles like atoms, but incredibly small, vibrating strings. Imagine these strings like tiny rubber bands, and their vibrations determine the type of particle they become. This theory aims to explain how all the forces and particles in the universe are connected, potentially providing a single framework to understand everything from the smallest particles to the biggest galaxies.



String Theory

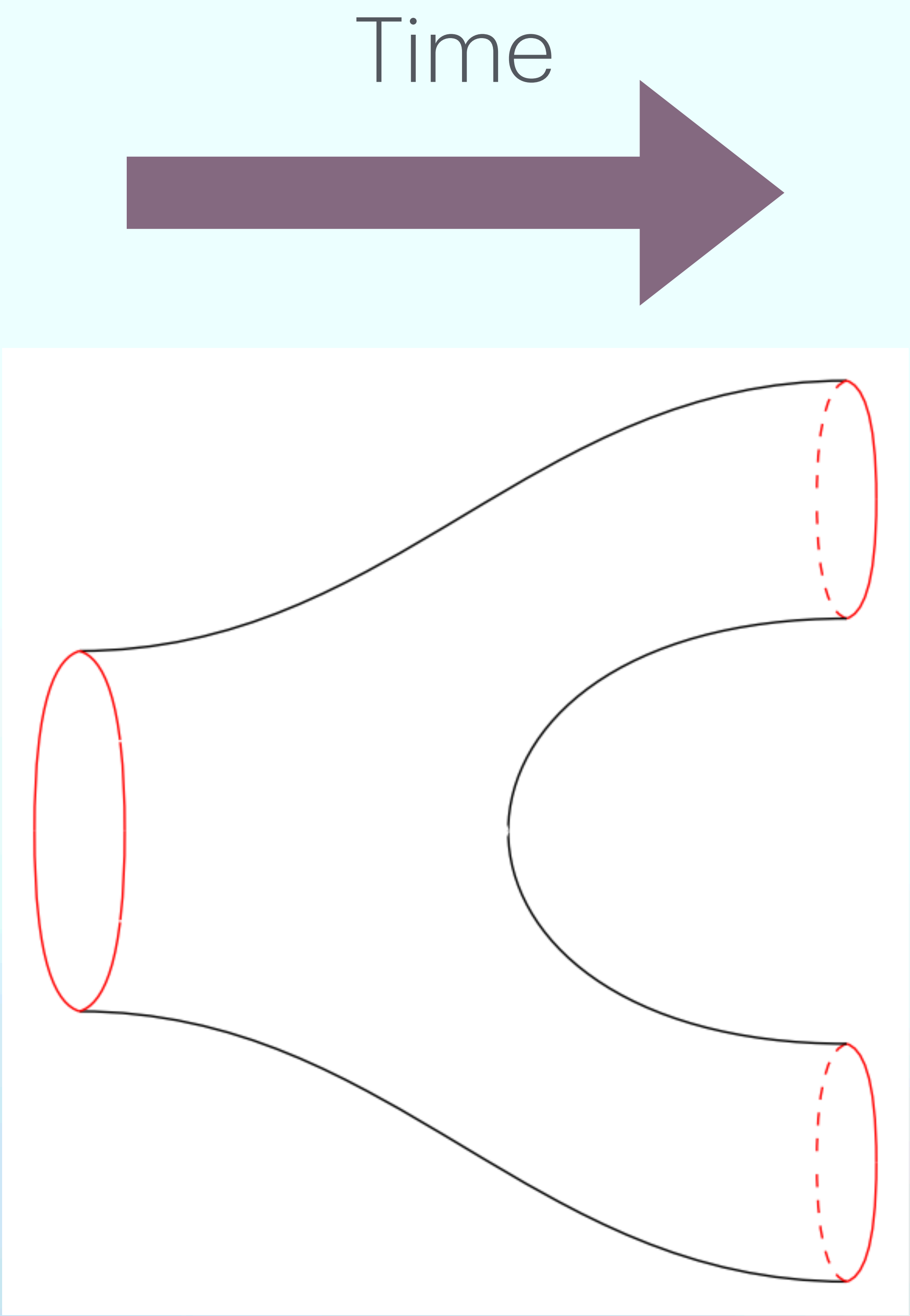
How strings evolve in time

- Imagine a closed string as a circle.

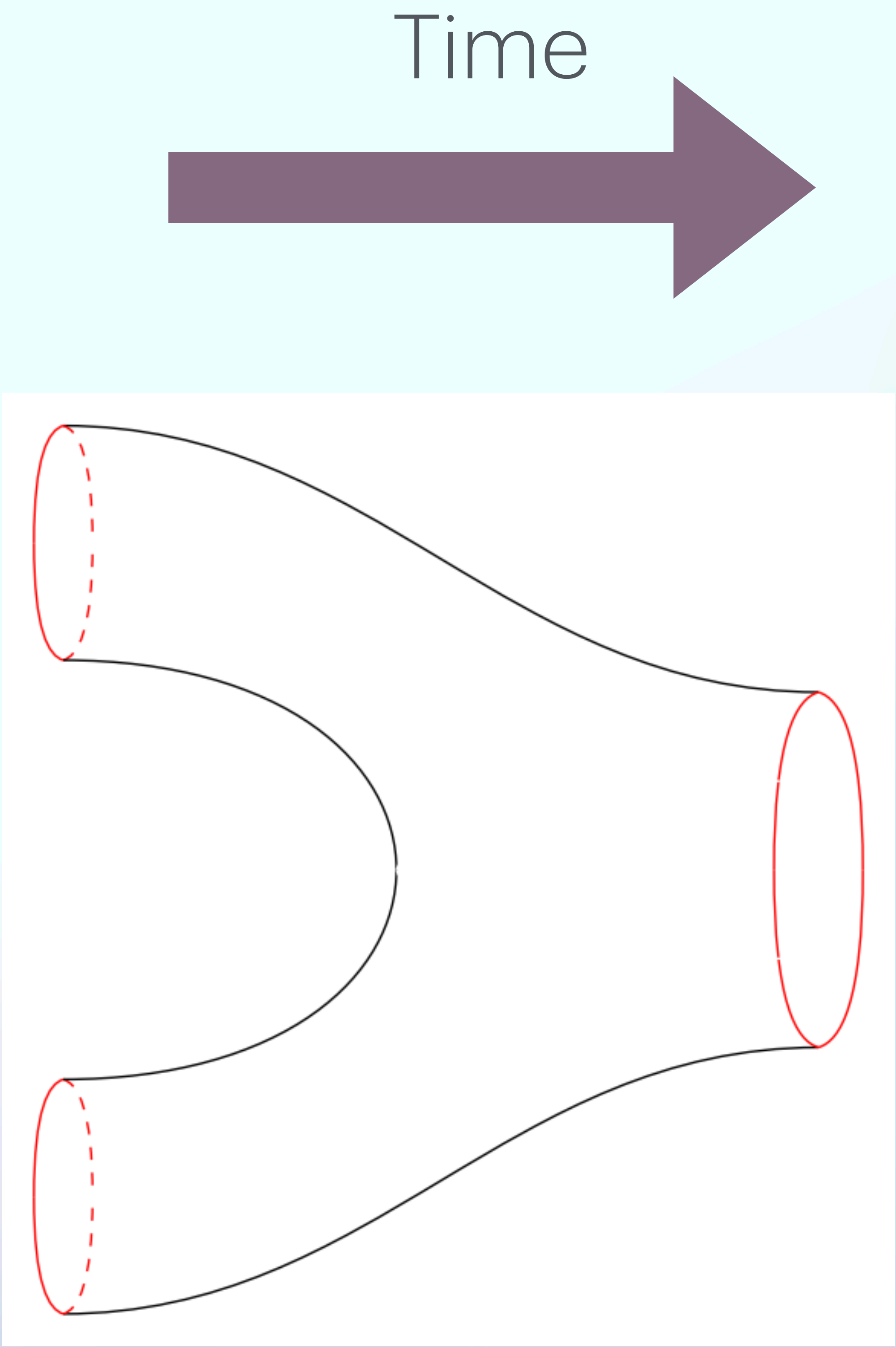


String Theory

String can divide and merge



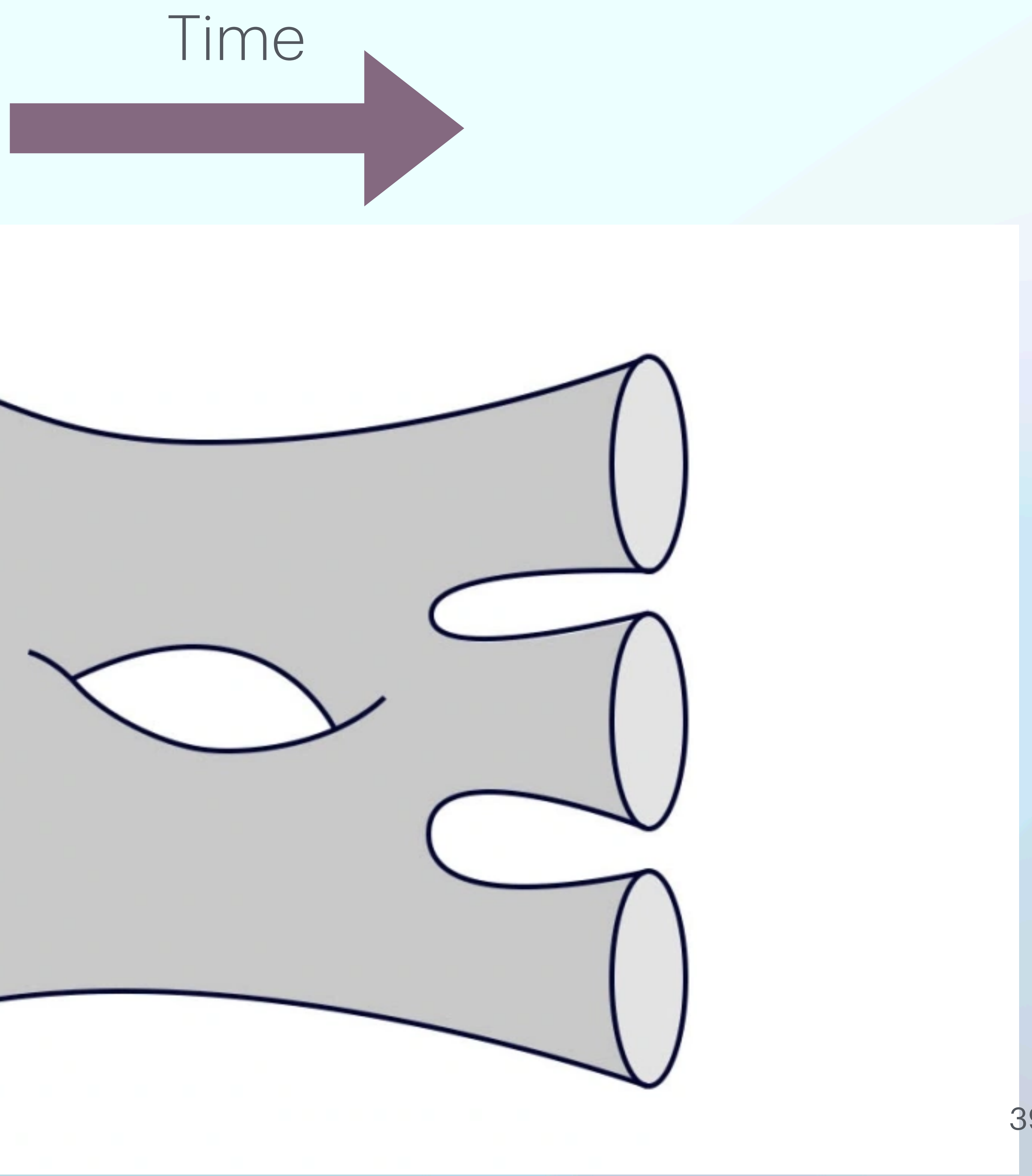
Dividing String



Merging Strings

String Theory

Time Evolution of a String



- As strings evolve in time, they divide and merge, creating a surface with holes and openings.

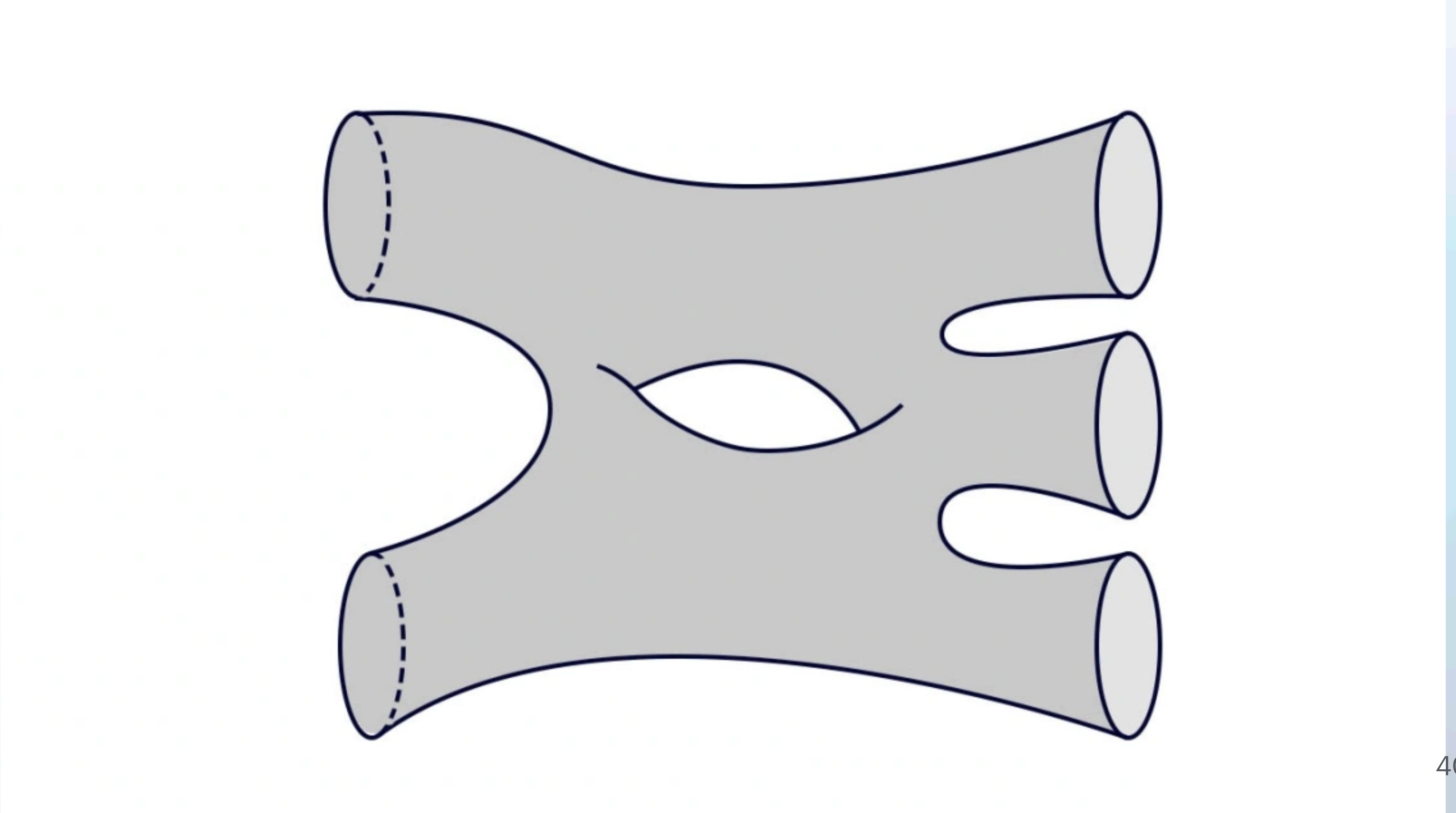
String Theory

Time Evolution of a String

- The number of holes g is called the **genus** of the surface.

- The number of initial plus final strings n is called the **boundary components**

$$g = 1, n = 5$$



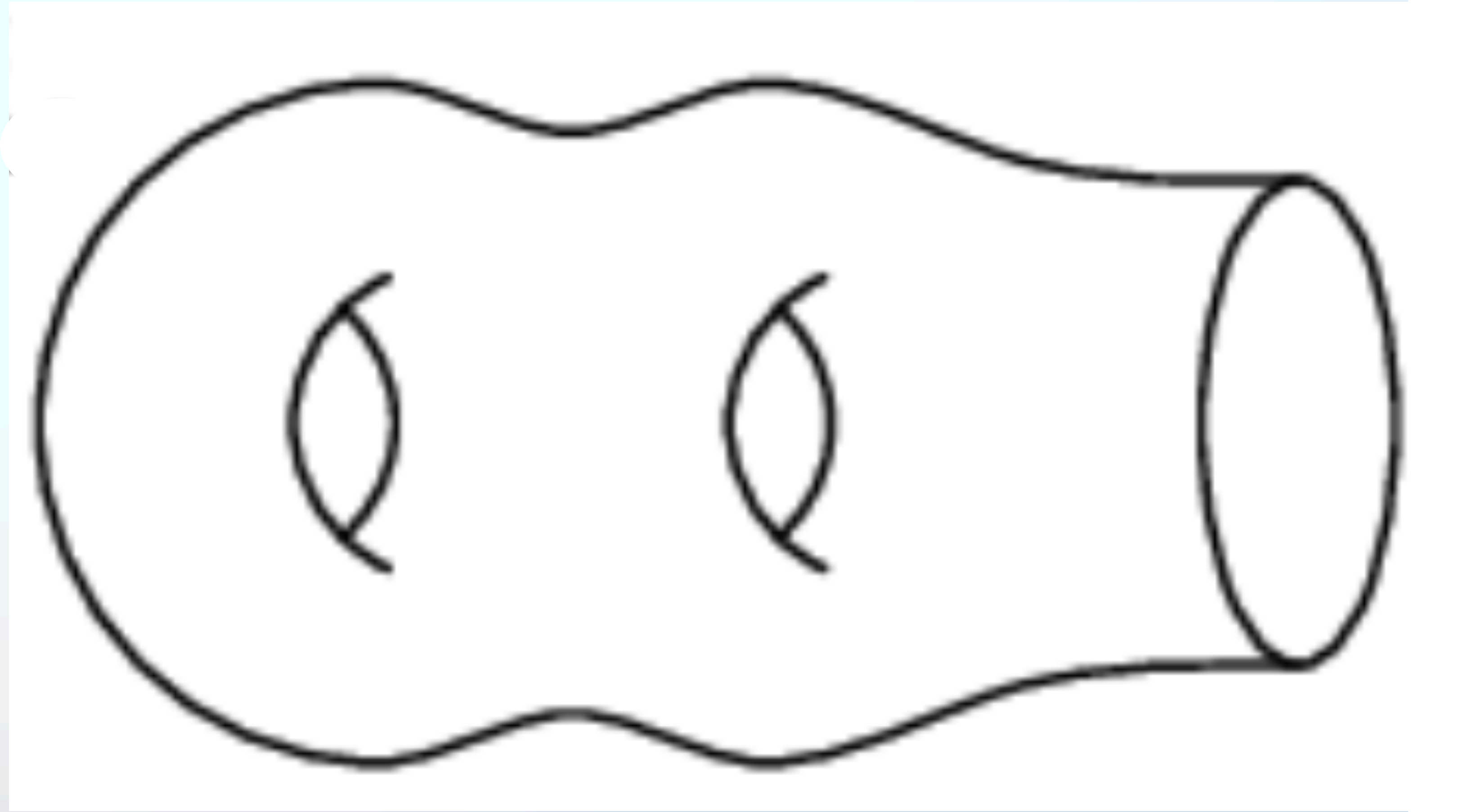
String Theory

Time Evolution of a String

- The number of holes g is called the **genus** of the surface.

- The number of initial plus final strings n is called the **boundary components**

$$g = 2, n = 1$$



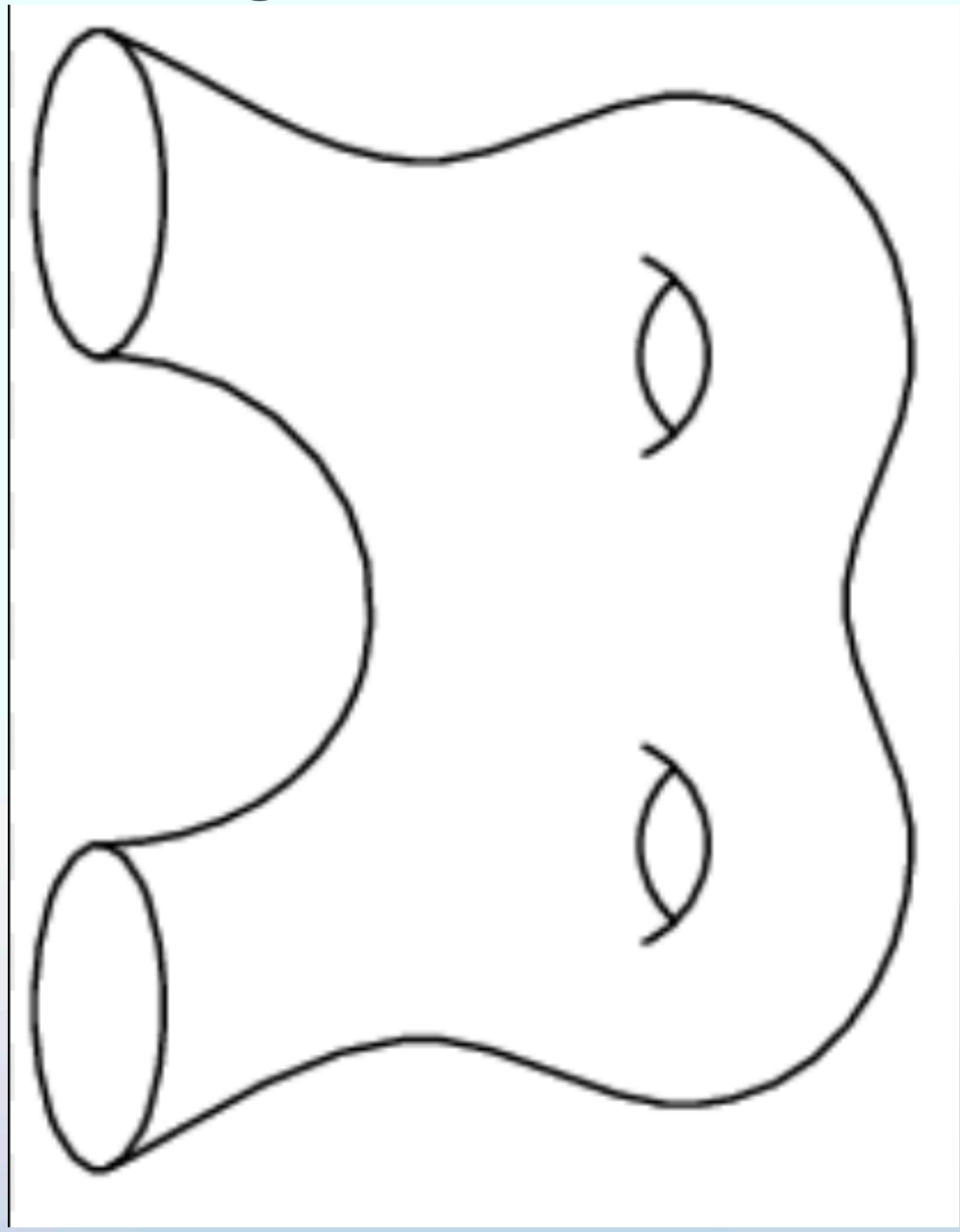
String Theory

Time Evolution of a String

- The number of holes g is called the **genus** of the surface.

- The number of initial plus final strings n is called the **boundary components**

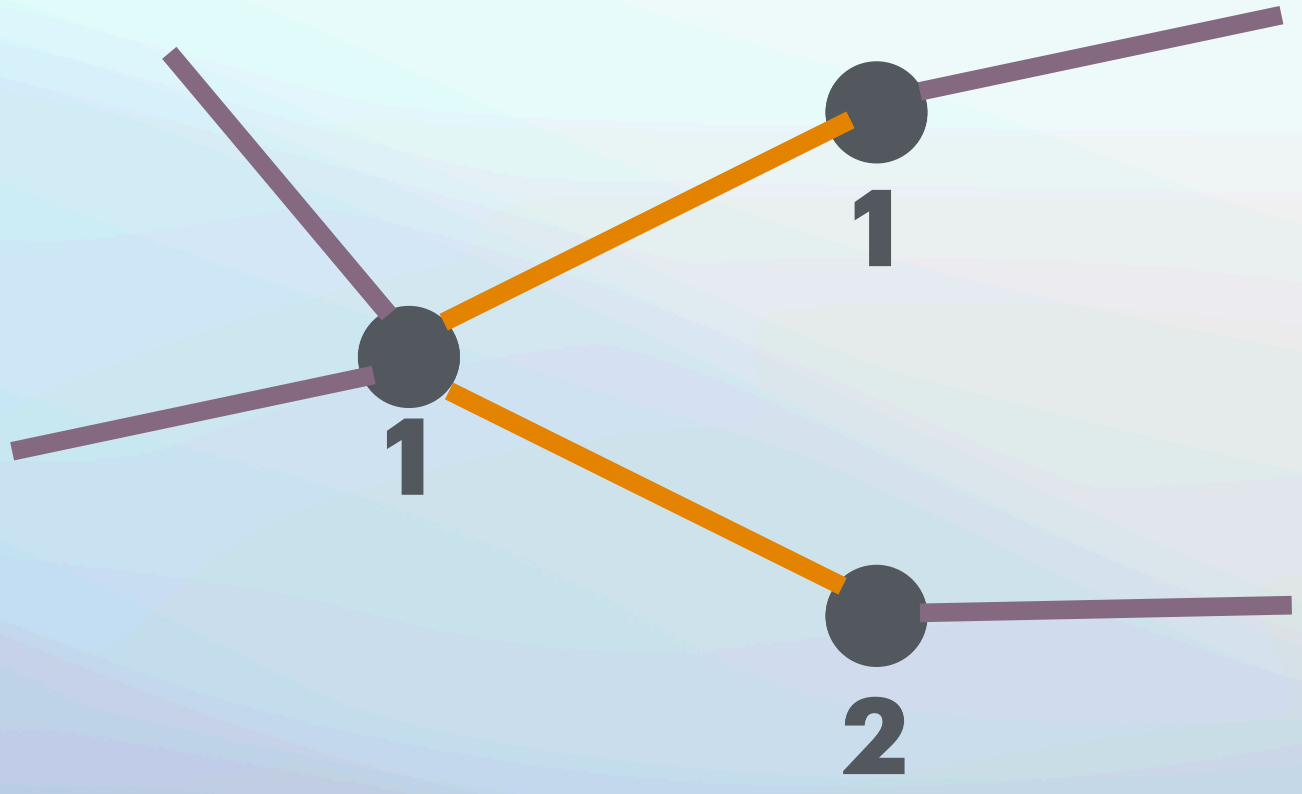
$g = 2, n = 2$



String Theory

Feynman Diagrams

- Feynman expansions list all possible ways strings can evolve generating the same surface. That means the same genus, and number of boundaries.
- To simplify the job, these surfaces are represented as **graphs**.
- A **graph** is a set of vertices connected by edges. Each vertex has an associated number. Some edges are connected just to a single vertex. These are called **free** edges.



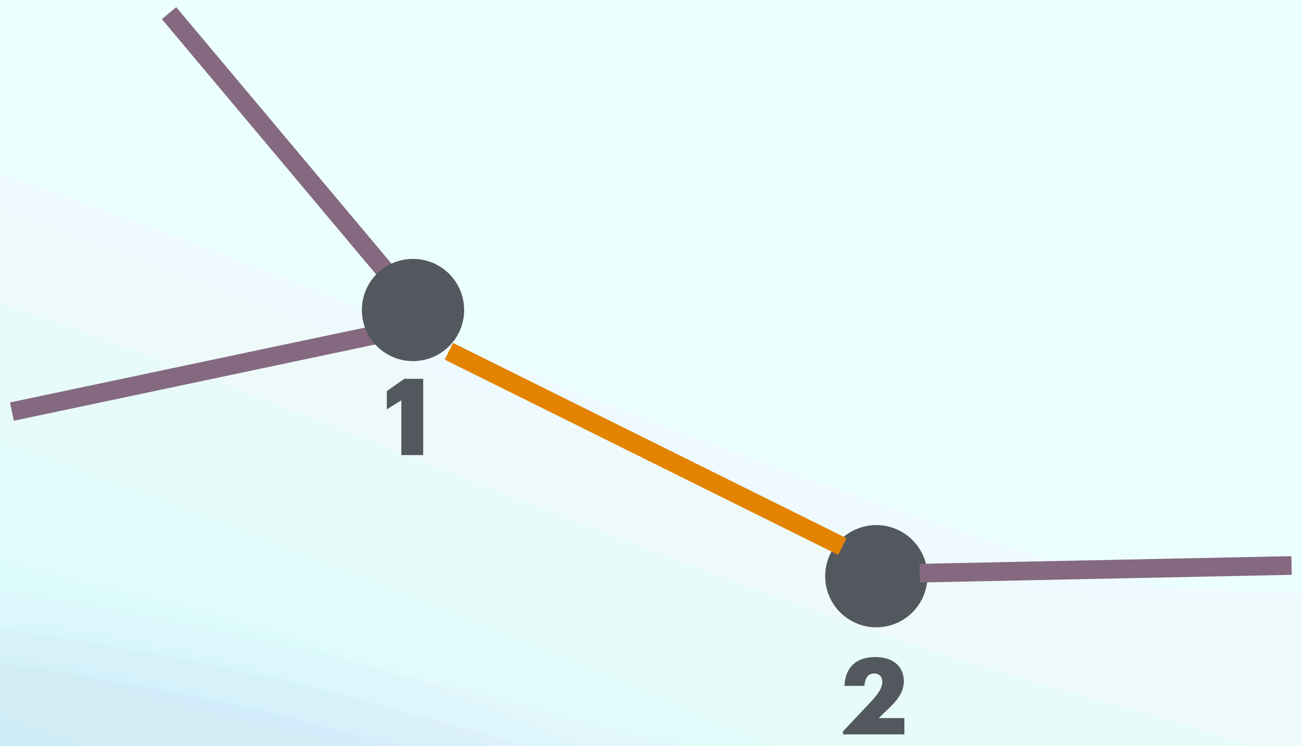
Regular Edges

Free Edges

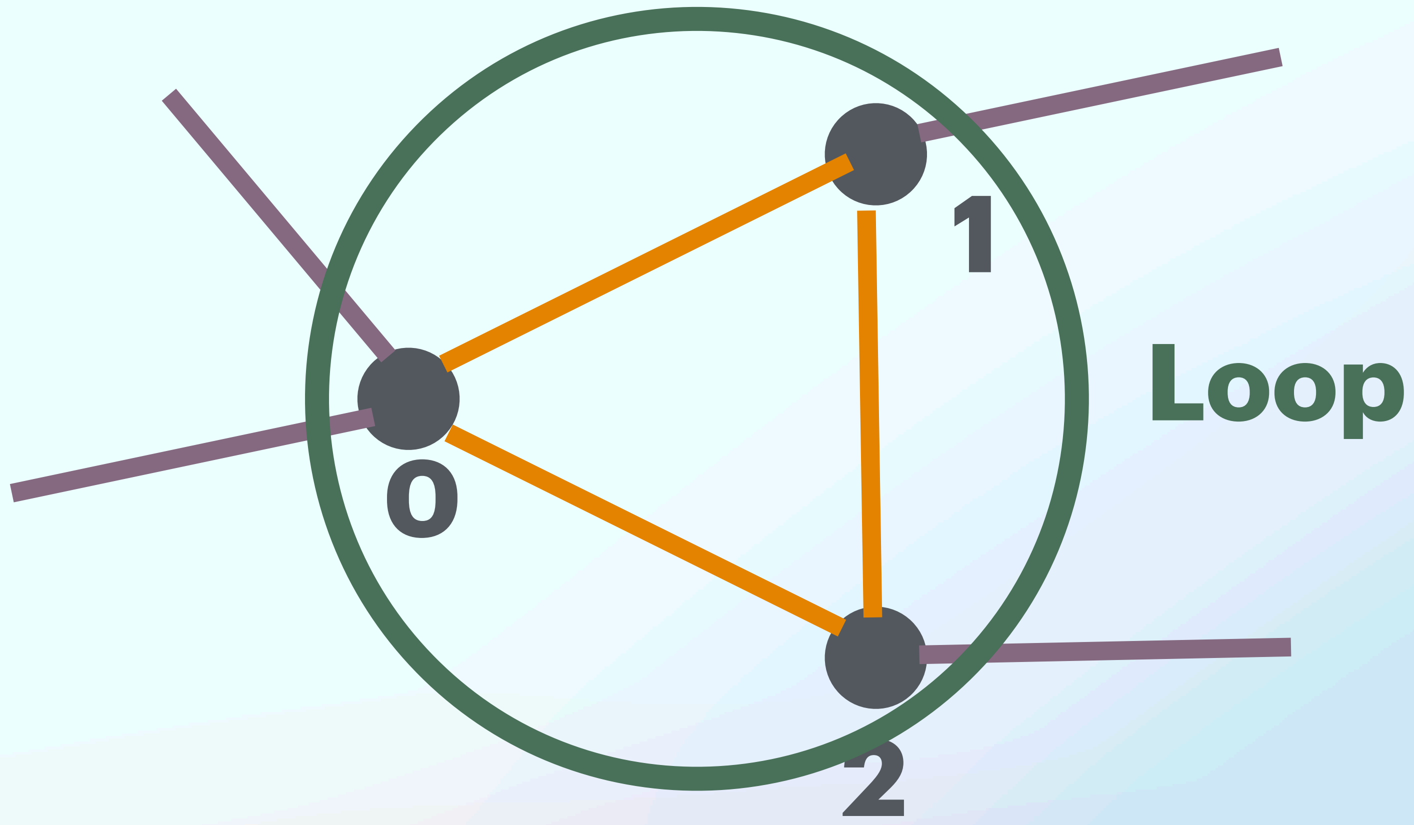
String Theory

Feynman Diagrams

- The **genus** of a graph is the sum of the values at all vertices, **plus** the number of loops



$$g = 1 + 2$$

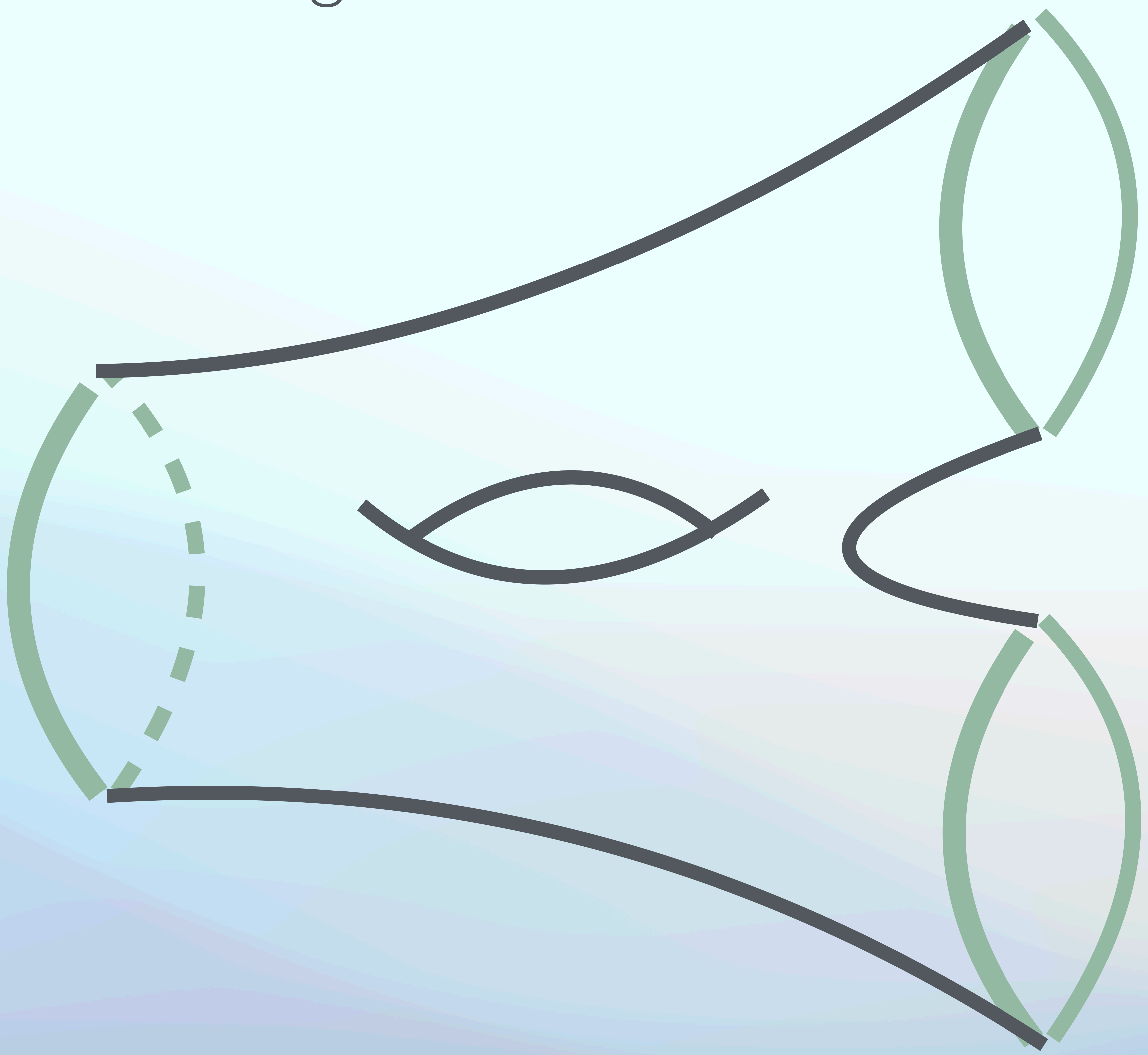


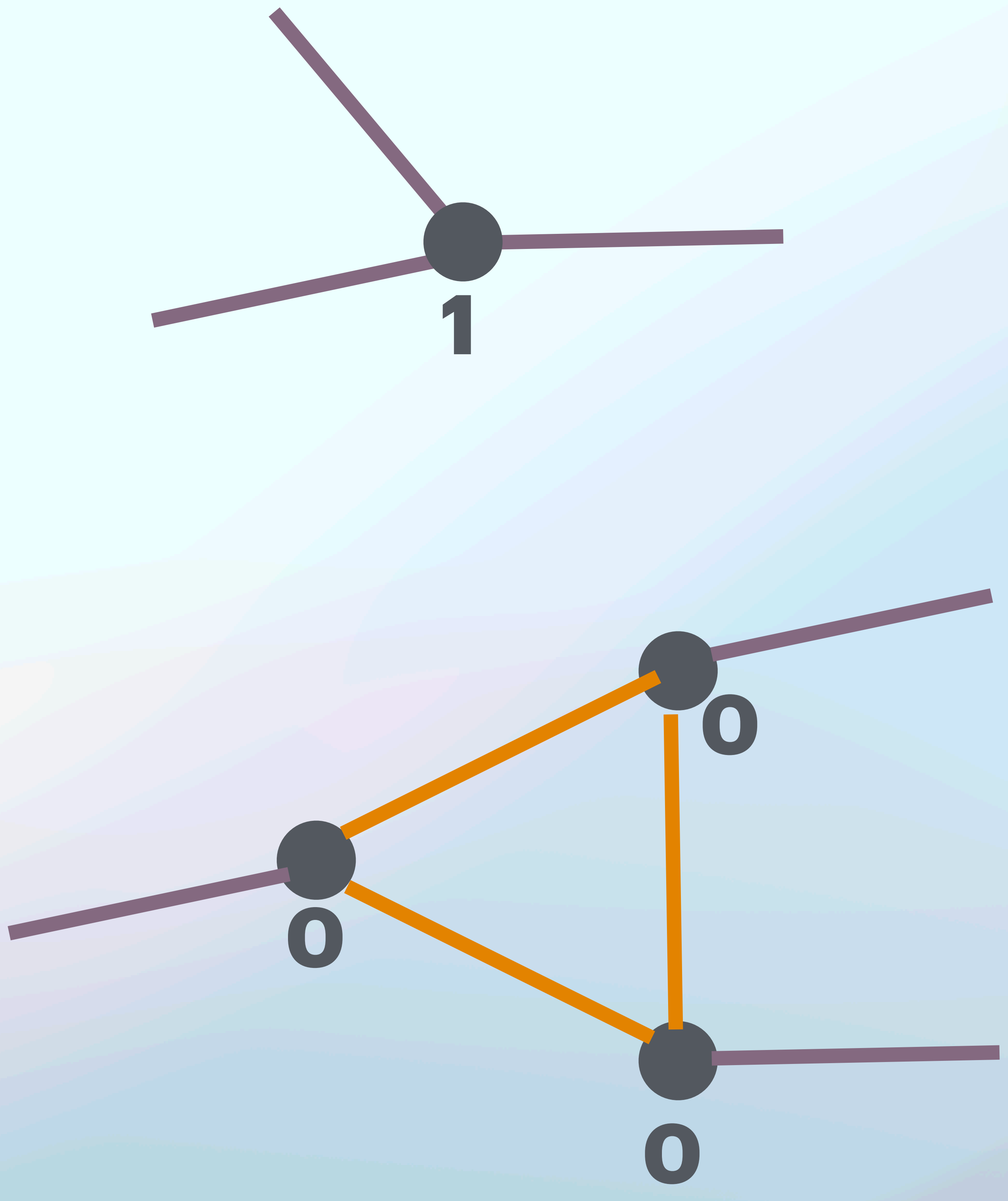
$$g = 0 + 2 + 1 + 1$$

String Theory

Translating between surfaces and graphs

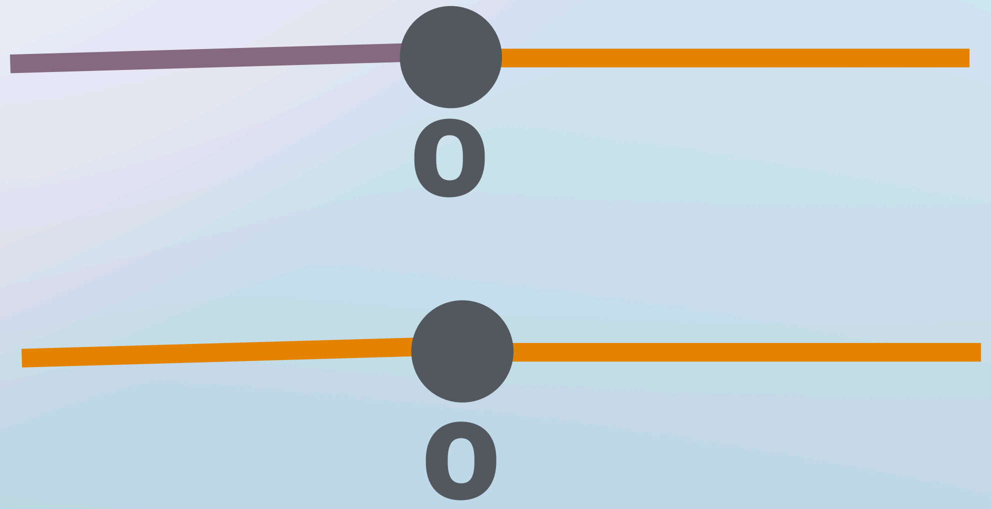
- A surface of genus g and n boundaries, can be represented by several graphs of genus g and n free edges.



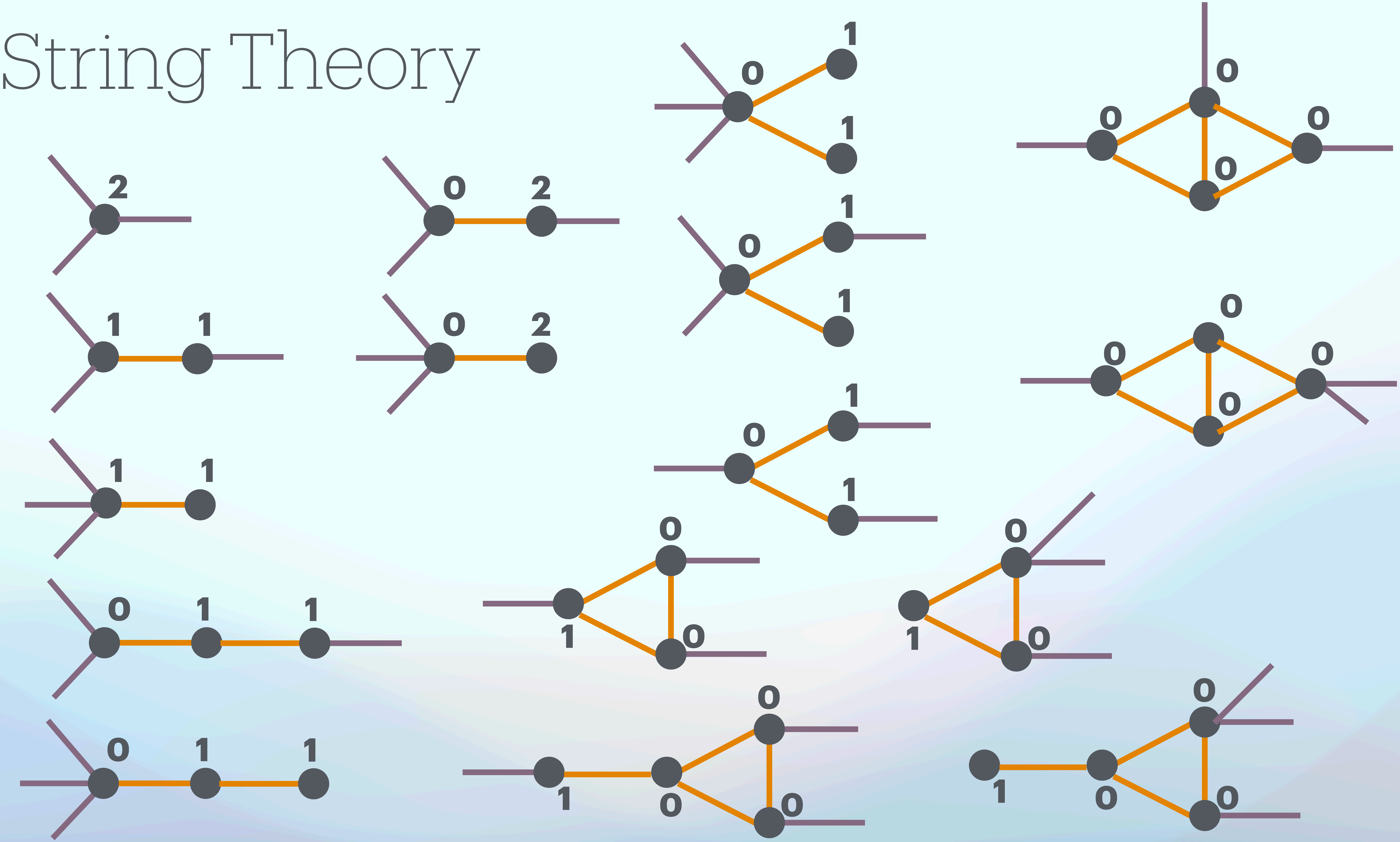


Find all graphs corresponding to this surface

This is **NOT** allowed



String Theory



Importance of Feynman Expansions

- Feynman graph expansion provides a powerful framework for systematically calculating particle interaction probabilities in quantum field theories. By summing over all relevant diagrams and using the corresponding Feynman rules, physicists can make precise predictions about the outcomes of particle interactions.

- Having a concrete graphical way to enumerate this is essential for the calculations.

Conclusion

- Enumerative Geometry problems are often “easy” to state, but hard to answer
- Deeply connected to combinatorics, graph theory, physics
- Still an active field of research (although many concepts are “ancient”)
- Email me if you want to learn more: **rguigo@math.harvard.edu**

Thanks for listening!